Pricing and Matching in the Sharing Economy

by

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Abstract

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We study operational problems related to the sharing economy. Sharing economy platforms such as Uber offer the crowdsourced suppliers a wage for providing services/goods, and charge the customers a price for using them. In Chapter 2, we study the fixed commission rate contract practiced by many sharing economy platforms. We show that by using the optimal flat-commission contract, the platform achieves at least 75% of the optimal profit of the first-best benchmark, in which the platform freely chooses the price and wage under various market conditions.

In Chapter 3, we consider a platform's problem of dynamically matching random demand and supply of heterogeneous types in a periodic-review fashion. The platform decides the optimal matching policy to maximize the total discounted rewards minus costs. We provide sufficient and robustly necessary conditions only on matching rewards such that the optimal matching policy follows a priority hierarchy among possible matching pairs.

In Chapter 4, we study the dynamic matching problem in Chapter 3 under two specific forms of reward structures. First, we consider the problem with horizontally differentiated supply and demand types. In that problem, supply and demand types locate on a unidirectional circle. The unit matching reward between a supply type j and a demand type i is a decreasing function with respect to the unidirectional distance from the location of j to that of i on the circle. We then study the problem with vertically differentiated supply and demand types, for which we impose a reward structure in which types have quality differences. For both cases, we apply the results in Chapter 3 to characterize the optimal matching policy.

In Chapter 5, we study the pricing behaviors of two agents under incentives generated from social comparison. We demonstrate how opposite-directional social comparisons interact with demand variability to change competitive behaviors. In particular, we show that the stronger the behind aversion behavior, the more intense the price competition, and that there is a threshold on the market variability above which price competition is more alleviated and below which price competition is more intensified, when the agents exhibit stronger ahead-seeking behavior.

Dedication

To my wife, my parents and my son

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Contents

Chapter 1

Introduction

In the past decade, we have witnessed the rise of the sharing economy. With the help of technological platforms, ordinary people are able to share their excess resources and receive monetary payment in return. As one of the pioneers in the recent rise of the sharing economy, the venture-funded startup Uber launched non-taxi ridesharing services that allow a prospective passenger to use a smartphone app to book a space in a nearby car owned by someone else. Ubers service has now expanded to over 300 cities in the world, recording its 2 billionth ride in 2016. Together with other startups such as Sidecar and Lyft, the company is leading a disruption to the taxi industry. Like the ridesharing services, the car rental company FlightCar sources from the crowd by offering travellers free parking at the airport and renting out their cars in exchange. The logistics industry has also benefited from the idea of the sharing economy. In addition to offering rides, Uber also has its drivers deliver small packages. Amazon's "Amazon Flex" program uses independent contractor drivers to deliver packages. Even the power industry is benefiting from the thriving of the sharing economy. KiWi power, a grid company in the UK, operates a virtual power plant which enables power capacity to be shared among users. It pays users for agreeing to switch off their appliances (e.g., when a freezer is cold enough) and sell the spare capacity to the national grid. Two other examples of the modern sharing economy are Airbnb, where people rent out lodging spaces in their homes, and TaskRabbit, which enables its users to outsource small tasks to people in the neighborhood.

Technological platforms such as Uber crowdsource supply from ordinary people, and their decision making revolves around effectively matching crowdsourced supply with customer demand. In this thesis, we study such a platform's problems on coordinating crowdsourced supply with demand through pricing and quantity matching decisions.

In Chapter 2, we consider a platform which crowdsources a service from independent suppliers and sells it to customers. The platform offers a wage to the supply side and charges a price to the demand side. We study the performance of the widely practiced, flat, across-the-board commission contracts, under which the platform takes a fixed cut and the wage is equal to a pre-determined fraction of the price. As a first-best benchmark, the platform can freely choose the price and wage under various market conditions. For a given realized market condition, we show that the joint price and wage optimization in the first-best benchmark can be reduced to one-dimensional problem of solving for the optimal matching quantity, and the optimal price has a U-shaped relationship with the wage. The latter implies that our two-sided pricing problem is in stark contrast with the traditional supply chain settings and the economics literature on the two-sided market. With the pre-committed commission, the platform has its hands tied in varying prices to match supply with demand, under supply and demand uncertainty. However, surprisingly, we show that a commission contract can be optimal or near-optimal for the platform and suppliers. In particular, as long as all possible supply curves are concave in the wage which seems consistent with 2015 US hourly wage data, by using the optimal flat-commission contract, the platform achieves at least 75% of the optimal profit of the first-best benchmark. We then make several extensions to other alternative platform objectives such as welfare maximization, to piecewise commission contracts, and to suppliers who set their own wages.

In Chapter 3, we consider an intermediary platform's problem of dynamically matching demand and supply of heterogeneous types in a periodic-review fashion. More specifically, there are two disjoint sets of demand and supply types. There is a reward associated with each possible matching of a demand type and a supply type. In each period, demand and supply of various types arrive in random quantities. The platform's problem is to decide on the optimal matching policy to maximize the total discounted rewards minus costs, given that unmatched demand and supply will incur waiting or holding costs, and will be carried over to the next period with abandonments. This problem applies to many emerging settings in the sharing economy and also includes many classic problems, e.g., assignment/transportation problems, as special cases. For this dynamic matching problem, we provide sufficient and robustly necessary conditions (which we call *modified Monge conditions*) only on matching rewards such that the optimal matching policy follows a priority hierarchy among possible matching pairs: if some pair of demand and supply types is not matched as much as possible, all pairs that have strictly lower priority down the hierarchy should not be matched. The modified Monge condition generalizes the Monge sequence condition, discovered by Gaspard Monge in 1781, which guarantees that a static and balanced transportation problem is solvable by a greedy algorithm.

In Chapter 4, we continue to study the dynamic matching problem with heterogeneous supply and demand types, but focus on more specific forms of reward structures. First, we consider the problem with horizontally differentiated supply and demand types. In that problem, types have (unidirectional) "taste" differences. More specifically, supply and demand types are assumed to locate on a unidirectional circle. The unit matching reward between a supply type j and a demand type i is a decreasing function with respect to the unidirectional distance from the location of j to that of i on the circle. We show that there exists a matching priority hierarchy related to "taste" locations: for any given demand (or supply) type, the closer its distance to a supply (or demand) type, the higher the priority to match the closer pair. Along the priority hierarchy, the optimal matching policy has a *match-down-to* structure for any pair of demand and supply types: there exist state-dependent thresholds; if the levels of demand and supply are higher than the thresholds, they should be matched down to the thresholds; otherwise, they should not be matched. We then study the problem with vertically differentiated supply and demand types, for which we impose a reward structure in which types have "quality" differences. For these vertically differentiated types, the optimal matching policy has an even simpler *top-down matching* structure (in short, "line up, match up"): line up demand types and supply types in descending order of their "quality" from high to low; match them from the top, down to some level. When demand and supply types have the same abandonment rate, the match-down-to levels have monotonicity properties with respect to the system state, and the one-step-ahead heuristic policy has a simplified state-dependent structure.

In Chapter 5, we study the pricing behaviors of two agents under incentives that generate social comparison. Independent suppliers in the sharing economy, as well as decision makers in other industries, often compete with each other for customers through pricing or other decisions. As human beings, those decision makers are subject to social comparison. In the chapter, the two agents sell differentiated substitutable products under additive demand uncertainty, and their decisions are influenced by social incentives. Social comparison theory, as well as conventional wisdom, suggests that social comparison behaviors, such as behind aversion (upward comparison) and ahead seeking (downward comparison), all work in the similar fashion to intensify competition. We demonstrate how opposite-directional social comparisons interact with demand variability to change competitive behaviors. In particular, we show that the stronger the behind aversion behavior, the more intense the price competition. Surprisingly, there is a threshold on the market variability above which price competition is more alleviated and below which price competition is more intensified, when the agents exhibit stronger ahead-seeking behavior. In addition, we also find that the agents' biased perceptions of market variability may reduce price competition as the social comparison effect is influenced in different ways by the agent's own market variability and by the apparent market variability of the competitor. These insights are robust under multiplicative demand uncertainty, but they are reversed for complementary products. We identify the driving forces behind these results.

Chapter 2

Optimal Price and Wage, and Fixed Commission

2.1 Introduction

Unlike traditional businesses, sharing economy platforms, which are essentially intermediary firms that connect the supply side with the demand side, often employ crowdsourced supply (physical goods or intangible services) to meet customer demand. For example, ride-hailing platforms such as Uber and Lyft rely on free-lance drivers who decide when and how long to work. Short-term rental platforms such as Airbnb crowdsource property listings from hosts who manages the availability of their properties on their own. Compared with sourcing from suppliers in regular business processes, crowdsourced supply from independent agents tends to be less costly but has a higher level of uncertainty on the supply side, due to lack of direct control by the platform. Just like price-sensitive customers on the demand side, crowdsourced suppliers are also sensitive to their rewards for providing services. Thus wage for suppliers and price for customers are the two key controls for the platform such as Uber and Lyft to coordinate supply and demand. Optimally determining both wage and price can be a nontrivial task. On the one hand, the platform needs to offer a decent wage to incentivize suppliers and a reasonable price to attract customers. On the other hand, an adequate profit margin (i.e., the gap between the price and wage) is required to ensure profitability. In addition, time-varying supply and demand conditions often add to the complexity of the platform's problem of deciding the optimal price and wage, which need to be adjusted over time according to the market conditions. For example, Uber implements the so-called "surge pricing" to contingently match supply with demand in each designated region.

In practice, it is common for the platform to charge a flat, across-the-board commission rate that applies to all market conditions. For example, Uber started its business taking a 20% commission on all rides, and now has raised and lowered that rate in different cities depending on the supply of drivers and

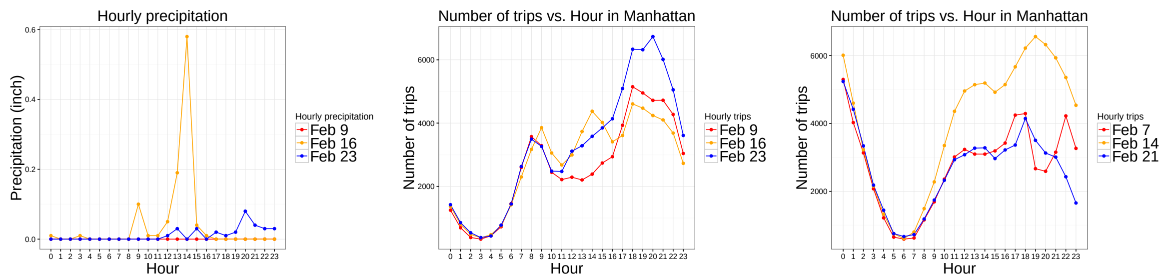


Figure 2.1: Demand Driven by Weather and Special Occassion

rider demand. Airbnb charges its hosts a 3% host service fee for each reservation. In a fixed commission contract, all parties agree on a specific rate, according to which the platform takes a portion of the total revenue generated through the sharing transactions. (The platform may adjust the fixed rate but keeps it for a relatively long period of time.) The major reason of such a practice is *legality*. It qualifies the platform to receive a brokerage license, as the intermediary business makes supply meet demand without actually buying and selling like retailers. Another appealing feature of this contract is its *simplicity* to understand and implement for both sides of the market. However, such a pre-committed crowdsourcing contract seems not optimal at the operational level. As supply and demand conditions change over time, the platform would ideally want to update both price and wage accordingly to match supply with demand. With hands tied under the flat commission contract, the pricing decision and its associated wage determined by the preset affine commission contract is suboptimal. In this chapter, we study the following research questions. Is it possible that the commission contract is indeed optimal for the platform? If not, how good is this contract?

Depending on time and day, the platform possibly faces many scenarios of market conditions. For instance, see Figure 2.1.[1] On Feb 16, 2016, Tuesday, because there was a heavy rain in the early afternoon, a peak in the number of Uber trips in Manhattan occurred in addition to the normal two peaks of a day corresponding to the morning and evening rush. On Feb 23, 2016, Tuesday, it was drizzling starting at the noon and through the rest of the day, we see that the normal demand pattern on Tuesday with no rain (e.g., as on Feb 9, 2016, Tuesday) was amplified proportionally due to the rain. Other than weather, special occasions can also drive demand. On the Valentine's Day of 2016, which is a Sunday, demand surged during the day as compared to a normal Sunday such as Feb 21, 2016. The late-night peak in the demand on Feb 7, 2016, Sunday, is due to the Super Bowl.

In our model, we focus on a market with supply and demand uncertainty, by associating each possible market condition with some probability. Under each market condition, there are a pool of potential suppliers with heterogeneous opportunity costs and customers with heterogeneous valuations. For a given pair of price and wage, there are certain number of customers who are willing to make a purchase

[1] The plots are generated from the Uber data requested and obtained by fivethirtyeight.com from New York's Taxi & Limousine Commission under the Freedom of Information Law. In this data set, for each trip, only the information about the pickup location and time is available.

and certain number of suppliers who are willing to provide the service. In other words, there exist a supply curve and a demand curve for each market condition. Both the suppliers and customers enter the market after assessing their likelihood of being matched. For a given price and a wage, the platform matches the demand and the supply after they enter the market.

To study the performance of a fixed commission contract, we start with a benchmark model in which the firm can freely choose the price and wage for each market condition. That is, in the benchmark, contingent on the realization of a market condition, the platform jointly sets the price and wage to maximize its profit. We show that the platform's optimal price has a *U-shaped* relationship with the wage. This implies that campaigns to improve wages and benefits such as imposing a higher minimum wage for the independent agents in a two-sided market likely also benefit customers on the demand side. The intuition is that as the wage increases, more suppliers would like to provide the service, and as a result, the platform is compelled to squeeze its own margin by lowering the price to utilize the larger amount of supply, resulting in an increase in customer surplus. Moreover, we show that the benchmark model as a two-dimensional price and wage optimization problem can be reduced to a one-dimensional problem of finding the most desirable matching quantity for maximizing profit.

Then we compare the optimal commission contract that pre-specifies a linear relationship between the price and wage, with the benchmark where there is full freedom of choosing the wage and price, under uncertainty market conditions. We show that if both supply and demand are affine functions with common wage and price sensitivity across market conditions, a commission contract is indeed optimal. In particular, if the willingness-to-sell of independent suppliers and the willingness-to-pay of customers are uniformly distributed and the uncertainty comes from the potential market size of customers and suppliers, a fixed commission contract is optimal. In general, the fixed commission contract is not optimal. However, as long as the supply curve is a concave function of wage (this seems consistent with 2015 US hourly wage data), we show that by using the optimal fixed commission contract, the platform achieves at least 75% of the optimal profit of the benchmark. For the case where the supply curve is not concave but has bounded growth, we provide a primitive-dependent performance bound of the optimal commission contract. For instance, if the supply curve has an increasing rate slower than a cubic function, we show that the commission contract achieves at least 52.75% of the optimal profit of the benchmark. We further test performance of the commission contract numerically. The results show that the actual performance is better than the provable worst case performance bounds.

We extend our base model in three directions. First, we study objectives other than profit maximization. Like the base model, a benchmark in which the firm jointly optimizes price and wage can be reduced to a one-dimensional optimization problem that solves for the desirable matching quantity. Moreover, by including either supply side *or* demand side surplus taken into the its objective, the platform would increase the wage and lower the price, thereby benefiting *both* sides. When the platform's objective is to maximize the social welfare, including the platform's profit, the supply side and demand side surplus, the optimal wage becomes the highest, the optimal price becomes the lowest and the matching quantity becomes the largest. Second, we study piecewise commission contracts as opposed to the fixed commission contract. We identify the number of pieces needed to guarantee that the performance of the contract is different from the optimal performance of the benchmark by a given percentage. Third, motivated by Airbnb, we study a market in which the platform sets the fixed commission, but the individual suppliers decide the prices. We show that for a given commission, this price-setting case may lead to lower prices than the price-taking model, thus hurting the platform. However, by optimally setting the fixed commission in advance, the platform can mitigate the loss. Our numerical experiments show that the loss is not large after the fixed commission is optimized in advance. If there is only one market condition, the platform can completely eliminate the loss.

We make the following three main contributions. First, we show that the two-sided market with crowdsourced supply is fundamentally different from the traditional supply chain setting and the two-sided market literature in economics (see more discussion in the literature review). The difference may be hidden in a more sophisticated setting, e.g., a queueing formulation. By identifying the difference, we provide a theoretical justification for studying the two-sided matching models from the operational perspective where the matching quantity is naturally taken as the minimum of the supply and demand (analogous to the sales volume as the minimum of the capacity and demand in the operations literature). Second, we show that the two-dimensional price and wage optimization problem can be reduced to a one-dimensional problem that solves for the most desirable matching quantity, thereby making the problem significantly simpler. This insight is robust under different platform objectives. It can potentially serve as a guidance to solve more complicated two-sided pricing problems. For example, we expect that the same insight would hold for a dynamic two-sided pricing problem where unmatched supply and demand can be carried from one period to another. Third, we contribute to the supply chain contract literature by studying the commonly practiced fixed commission contract as a type of crowdsourcing supply contract. More specifically, we provide constant or easy-to-compute performance bounds for this fixed commission contract under supply and demand uncertainty.

2.2 Literature Review

Optimal pricing problems have been extensively studied in the revenue management literature. This stream of papers typically considers a fixed supply side and price-sensitive demand. In contrast, we consider a supply side sensitive to the wage offered by the platform.

In economics, research on two-sided markets has studied platforms such as credit cards, video game consoles, and organ allocation/exchange. Rochet and Tirole (2003) consider a general model of competition between two platforms with the transaction volume in the *multiplicative* form of demand and supply. Such a form is reasonable for a two-sided market platform with a long-term goal. For instance, the credit card company cares about the potential transaction volume in proportion to $M \cdot N$ if there are N customers on the demand side using the credit card and M merchants on the supply side accepting the credit card for payment. For an overview on this stream of research, we refer the readers to Rochet and Tirole (2006), who also build a model with usage and membership externalities, in addition to providing a roadmap to the literature. Our paper studies the day-to-day pricing decisions of an on-demand matching platform that adapt to the changing market conditions, rather than equilibrium analysis and impact of network externalities. To that end, we model the transaction volume as the *minimum* of demand and supply quantities. For instance, if in a time period, there are M drivers and N riders within a neighborhood, the transaction volume is $\min\{M, N\}$. Such a form of transaction volume is better suited for an on-demand matching market and is most natural from an operational perspective.

Our paper is most closely related to the recent papers in operations management on price and/or wage optimization problems faced by on-demand service platforms, in which the retail price and/or the wage for the suppliers are optimized for maximizing the platform's profit, social welfare, etc. Taylor (2016) uses a queueing formulation to model an on-demand platform's business process. Under the assumption of a two-point distribution for both customers' and suppliers' valuations, the paper derives the demand rate and number of available suppliers in equilibrium for a given price and wage, and then maximizes the platform's profit by optimizing the price and wage. Similarly, Bai et al. (2016) also consider a queueing model for an on-demand platform, with suppliers' and customers' valuations following general distributions. The authors examine how the parameters, such as the available amount of supply, customers' waiting cost, and service rate, affect the optimal price and wage as well as the system performance. Banerjee et al. (2015) model the ride-hailing problem as a queueing network and study its fluid approximation. It is shown that under the large market limit, dynamic pricing with prices instantaneously reacting to supply-demand imbalances does not provide more benefit than the optimal static pricing. Chen and Hu (2016) study a dynamic market where sequentially arriving suppliers and customers may wait strategically for better prices. The authors show that a waiting-adjusted fixed pricing heuristic offered by the intermediary platform can deter strategic behavior and is close to optimal for a thick market. While those two papers focus on the near-optimality of fixed pricing, Cachon et al. (2016) study the benefit of dynamic pricing. The authors develop a model with price-sensitive self-scheduling suppliers, and compare the optimal contract (with dynamic wage and price), dynamic wage contract, dynamic price contract and the commission contract. They show numerically that the commission contract is near optimal, and that dynamic pricing is beneficial compared to static pricing. The previous papers do not consider the spatial element in on-demand matching systems. In contrast, Bimpikis et al. (2016) consider a spatial pricing problem on a ridesharing network of multiple locations. The authors show that if the network is strictly balanced or a two-type network, the optimal pricing policy is implementable by means of a uniform commission rate for all locations. In general, however, they demonstrate that the fixed commission rate may result in significantly lower profits, particularly in the presence of heterogeneity among the demand patterns in different locations. Gurvich et al. (2015) also study a service platform with self-scheduling suppliers. They control the maximum number of independent agents allowed to work in a period and the compensation for the agents, and show that self-scheduling leads to lower profit for the firm and lower service level for customers. Benjaafar et al. (2015) consider a peer-to-peer sharing model in which the population is heterogeneous in terms of the usage of a product. Through equilibrium analysis, the authors show that low-usage individuals choose to be renters and high-usage individuals choose to be owners. They also study the impact of the model parameters (such as price and commission) on the equilibrium outcomes (such as the level of ownership), as well as the platform's optimization problem for maximizing its profit or the social welfare (with respect to the rental price, for a given ratio of commission). While our paper also studies the platform's problem of optimizing wage and price, in contrast to the aforementioned papers, we derive structural properties of the optimal price and wage, and provide near-optimality bounds for the fixed commission contract (or prove its optimality in some cases) under *market condition uncertainty*.

In addition to price and wage optimization problems faced by on-demand service platforms, operations researchers have also studied other operational issues related to two-sided markets in the sharing economy. Allon et al. (2012) consider a moderating service platform and show that operational efficiency achieved by virtually pooling the agents together can harm the overall efficiency of the marketplace. Arnosti et al. (2015) show that in a matching market, reduced search cost can decrease aggregate welfare and that limiting the visibility of one side of the market can improve the welfare on both sides. Ibrahim (2015) studies a queueing model with random number of servers, which can happen in the presence of self-scheduling servers. Other related papers include bike sharing problems (e.g., Kabra et al. 2016 study the impact of bike accessibility and availability on ridership; Henderson et al. 2016 and Shu et al. 2013 study bike rebalancing problems) and electric vehicle sharing (He et al. 2016).

We also mention the literation on crossing networks in finance, where orders from buyers and sellers arrive randomly. In contrast with sharing economy platforms and intermediary marketplaces, orders submitted by buyers and sellers to a crossing network are directly matched at exogenously given prices and are invisible to the overall market. We refer the readers to Afèche et al. (2014) for a queueing analysis on such a system and Iyer et al. (2016) for welfare implication of operating a crossing network alongside traditional "lit" markets.

Our paper is related to the supply chain coordinating contract theory. With the interests of different parties in the supply chain being misaligned, researchers have designed various incentive-compatible contracts to achieve optimal performance of the entire supply chain. Examples include the two-part tariff contract (see, e.g., Jeuland and Shugan 1983), buy-back contract (Pasternack 1985), channel rebates (Taylor 2002), revenue sharing (Cachon and Lariviere 2005), etc. In our paper, for a given market condition, a fee for suppliers can coordinate the platform and the suppliers, while imposing fees for both suppliers and customers can induce the platform to set prices to achieve the optimal social welfare. With random market conditions, we study the fixed commission contract. Rather than considering the optimal supply chain performance, we compare the fixed commission contract with the optimal contract that maximizes the platform's profit, and show that it is near optimal for the platform. The commission rate in the fixed commission contract may be reminiscent of the revenue sharing parameter in the revenue sharing contract. However, they are fundamentally different. Coupled with the wholesale price, a continuum of revenue sharing parameters can achieve the same total supply chain surplus and arbitrarily allocate the coordination benefit between the supplier and retailer. In contrast, varying the commission rate affects the incentive of suppliers and the platform, resulting in different platform's profit, supplier and consumer surplus and total social welfare.

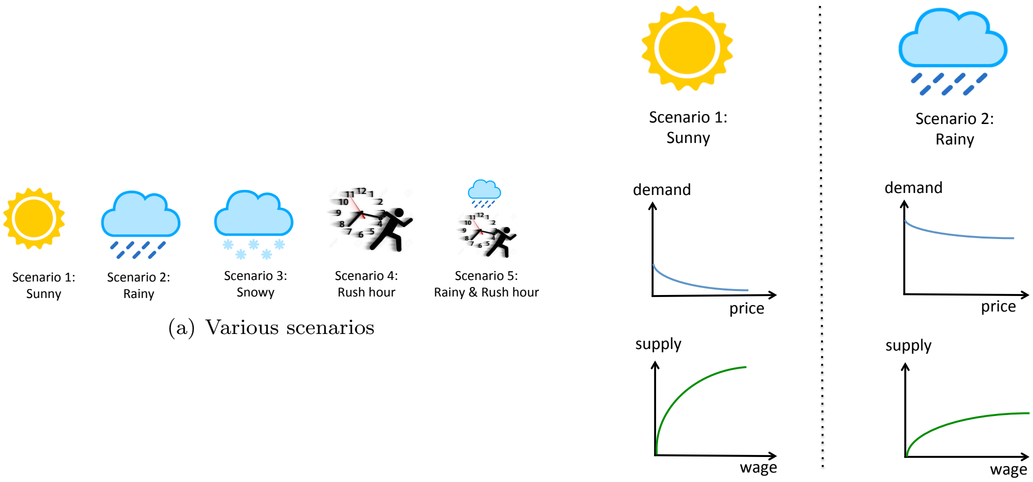
Finally, our work is also related to the robust pricing literature. For example, Bergemann and Schlag (2011) characterize the optimal pricing policy for the maximin expected utility and minimax regret problems, where the true demand distribution is known to be within a neighborhood of a model distribution. Cohen et al. (2015) propose to set the price as if the true demand function is linear and show that the resulting policy achieves good performance bounds for many common demand functions. In this paper, rather than the monopoly pricing problem, we consider the wage and price optimization problem with a two-sided market structure. We show that a pre-committed commission, coupled with contingent pricing, is "robust" in the sense that it achieve a guaranteed portion of the optimal profit for the platform, for any demand function and a broad range of supply functions.

2.3 The Model Setup

Consider a platform that coordinates the matching of customer demand for a service with crowdsourced supply. We will constantly resort to the ride-hailing service platform, e.g., Uber, as an example. Nevertheless, our model is applicable to other on-demand service platforms. Depending on time and day, the platform possibly faces many market conditions. For example, totally different market conditions of supply and demand may occur at rush hour and off-peak traffic times; in the same time period on a weekday as opposed to a weekend; or on the same day but in sunny as opposed to rainy weather. Let $\mathcal{K} = \{1, 2, \ldots, K\}$ be a set of possible scenarios of market conditions; see Figure 2.2(a) for an illustration. We assume that Scenario k occurs with probability ρ_k.

Each scenario is characterized by a demand curve and a supply curve (see Figure 2.2(b)). In Scenario k, the total amount of customers who are willing to pay for the service at price p is $d_k(p)$. We call $d_k(p)$ the raw demand function which naturally should satisfy the downward-sloping property; namely, $d_k(p)$ is a *decreasing* function in p.[2] As is consistent with the operations literature, the amount of satisfied customers may be capped by the available supply. Given a posted wage w, the total amount of independent contractors or suppliers, who are willing to provide the service, is $s_k(w)$. This is the

[2] The monotonicity in this paper is in its weaker sense unless otherwise specified.



(a) Various scenarios

(b) Demand and supply curves of scenarios

Figure 2.2: Scenarios of Market Conditions

amount of suppliers who would show up if they were guaranteed to be matched with a customer. We call $s_k(w)$ the raw supply function, which is assumed to be increasing in w. For simplicity, we assume $d_k(p)$ and $s_k(w)$ are continuous functions.

Sharing economy platforms often commit to a commission contract that determines a one-to-one relationship between what the customers pay and what the suppliers get. We define a general commission contract as follows.

Definition 2.1 (Commission Contract) *A commission contract $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a scenario-independent payment schedule, under which for any scenario, the wage offered to the suppliers is $w = f(p)$ if the price for customers is p.*

We see that the general commission contract is a payment schedule determined by the platform and enforced for all scenarios. In Stage 0, the platform can decide and commit to a commission contract (Figure 2.3).

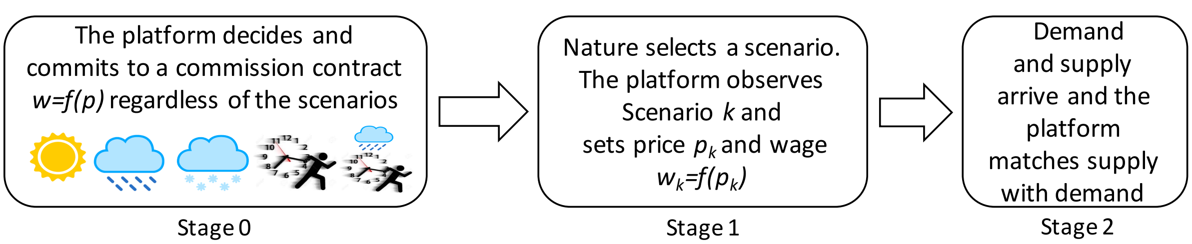


Figure 2.3: Sequence of Events under Commission Contract

In Stage 1, the nature selects a scenario. Given a realized Scenario k, the platform decides on the price p_k, with the wage following from the pre-determined commission contract as $w_k = f(p_k)$.

In Stage 2, all customers and suppliers observe p and w, and play a simultaneous game by deciding on whether to enter the marketplace. The platform clears the market by matching arriving demand and supply. If there are more suppliers than customers in the marketplace, suppliers are rationed to be matched with a customer, and vice versa. The rationing rule can be arbitrary and is announced upfront.

The fixed commission contract is a special form of the general commission contract under which the wage is a fixed portion of the price.

In the benchmark we consider, the sequence of events is as follows (Figure 2.4). In Stage 1, the nature selects a scenario and the platform freely sets price p and wage w. This is in contrast to the fixed commission contract, which requires the wage to be a fixed portion of the price. Hence, we call this model "the benchmark." In Stage 2, customers and suppliers decide whether to enter the marketplace and are matched by the platform if they do.

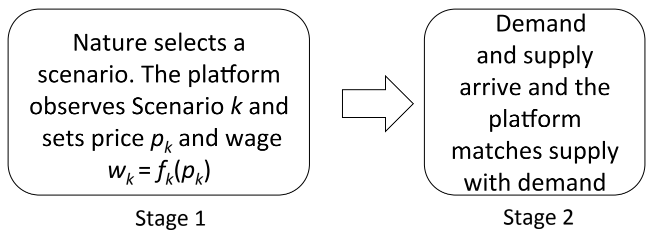


Figure 2.4: Sequence of Events: Benchmark

2.3.1 Demand and Supply Formation

We now characterize the behaviors of customers and suppliers in Scenario k, given the observed price p and wage w ($w = f(p)$ if a commission contract f is used). We allow customers or suppliers to anticipate their chances of being matched and assume that they know the demand curve $d_k(p)$ and supply curve $s_k(w)$. Somewhat surprisingly, the following result shows that the matching quantity in equilibrium can be computed as if the customers or suppliers are naive in the sense that they do not anticipate the likelihood of being matched.

Proposition 2.1 *For a given price p and wage w, the matching quantity when the suppliers and customers strategically anticipate the matching likelihood, is equal to* $\min\{d_k(p), s_k(w)\}$.

Proof of Proposition 2.1. We first prove the result if customers are naive without taking into account the chance of being matched but suppliers are strategically anticipating the chance. The case that both sides are strategic can be proved similarly.

We consider continua of customers and suppliers. The total mass of customers arriving at the platform is $d_k(p)$. Each supplier who is willing to provide the service at wage w either chooses to show up or not to. Hence, there always exists a mix-strategy equilibrium. Let $\tilde{s}_k(p, w)$ ($\leq s_k(w)$) be the total mass of suppliers who show up at the platform in equilibrium of a game where suppliers simultaneously decide whether to show up. According to the laws of large numbers for a continuum of random variables, $\tilde{s}_k(p, w) = \int S(t)dt$ is a deterministic number, where $S(t)$ is the binary random variable indicating whether supplier t shows up (see Uhlig 1996 who defines the integral under mean square convergence, or Al-Najjaar 2004 who considers a continuum-like discrete set of random variables).

If $s_k(w) \leq d_k(p)$, all suppliers anticipate that if they show up all of them will be matched. Thus, it is a dominant strategy for each supplier to show up and that all suppliers show up is an equilibrium with no other possible equilibrium. As a result, $\tilde{s}_k(p, w) = s_k(w)$ and the matching quantity is $\min\{d_k(p), \tilde{s}_k(p, w)\} = \min\{d_k(p), s_k(w)\} = s_k(w)$, where the last equality is due to $s_k(w) \leq d_k(p)$.

If $s_k(w) > d_k(p)$, not necessarily all of the suppliers would show up in equilibrium, because if they did, some of them would be rationed without getting a task and the expected earning is lower than the perceived wage w. Thus, $\tilde{s}_k(p, w) \leq s_k(w)$. Moreover, $\tilde{s}_k(p, w)$ cannot be lower than $d_k(p)$; otherwise if $\tilde{s}_k(p, w) < d_k(p)$, it is strictly beneficial for a positive mass of suppliers who choose not always to show up to deviate by arriving at the marketplace, which is a contradiction. As a result, we have $\tilde{s}_k(p, w) \in [d_k(p), s_k(w)]$ under any mixed-strategy equilibrium. Thus, the matching quantity is $d_k(p) \wedge \tilde{s}_k(p, w) = d_k(p) = d_k(p) \wedge s_k(w)$, where the last equality is due to $s_k(w) > d_k(p)$. □

With Proposition 2.1, we can assume that the suppliers are "naive" without loss of generality. The intuition is as follows. For a given price p and wage w, the potential demand who accepts the price is $d_k(p)$ and the potential supply who accepts the wage is $s_k(w)$. If one side, say the demand side, is in short, agents in the supply side may not always show up, as they need to take into account the chance of being matched. But they would not lower the chances of showing up such that the supply in the market is lower than the potential demand. There may be multiple mixed equilibria, but they all share the same feature that the matching quantity is determined by the side that is in short.

One can also interpret our stylized model as a snapshot of the fluid counterpart of a stochastic system. For example, in Scenario k, supply and demand arrive at the market following a Poisson process with supplier arrival rate $s_k(p)$ and demand arrival rate $d_k(p)$, respectively, over a certain period time. Assuming suppliers and customers are sufficiently patient not to abandon the wait, the throughput rate (i.e., the rate at which matched supply and demand pair leave the system) is $\min\{d_k(p), s_k(w)\}$. In the base model, we assume the platform's objective is to maximize its own profit. In the extensions, we study and compare the alternative objectives such as maximizing the platform and suppliers' surplus and maximizing the total social welfare.

2.4 The Benchmark

In the benchmark, the platform sets and announces the price p and wage w, *contingent* on the realized scenario and in anticipation of the market formation as the outcome of the announced price and wage.

This is consistent with the "surge pricing" practice of the ride-hailing platforms such as Uber and Lyft. We also assume that $s_k(0) = 0$ (i.e., no supplier would be willing to join the market if the wage is 0), and that $\lim_{p\to\infty} d_k(p) = 0$ (i.e., demand would be choked off if the price is set outrageously high). By Proposition 2.1, the profit of the platform in any Scenario k given price p and wage w, $\pi_k(p, w)$, is the product of the profit margin $p - w$ and the total matched amount of supply and demand, i.e., $\pi_k(p, w) = (p - w) \min\{d_k(p), s_k(w)\}$.

For notation convenience, in the rest of this section, we focus on one individual scenario, and thus omit the subscript k for the supply and demand functions. Given the profit margin $p - w$ and the total supply and demand matched in the scenario, the profit of the platform is

$$\pi(p, w) = (p - w) \min\{s(w), d(p)\}. \tag{2.1}$$

We first investigate how the optimal price changes as a function of exogenously given wage. The result demonstrates how the platform's profit maximization problem is different from that of a retailer in a traditional supply chain setting as $\max_p (p - w)d(p)$ or that of an platform in a two-sided matching market formulated in the economics literature as $\max_{p,w}(p - w)s(w)d(p)$. Then, we proceed to solve the joint price and wage optimization problem by reducing it to a one-dimension problem.

2.4.1 Optimal Price as a Function of Exogenous Wage.

For a given wage w, denote the *market clearing price*, by $p^c(w) \equiv \inf\{p \geq 0 \mid d(p) \leq s(w)\}$, denote the *revenue maximizing price*, by $p^m(w) \equiv \inf \arg\max_{p\geq 0}(p - w)d(p)$, and denote the platform's optimal price, by $p^*(w) = \inf \arg\max_{p\geq 0} \pi(p; w)$.

Theorem 2.1 (OPTIMAL PRICE AS A FUNCTION OF WAGE HAS A U-SHAPE) *Suppose* $\lim_{w\to 0} s(w) = s(0) = 0$, $\lim_{p\to p^c(0)} pd(p) = 0$ *and* $(p - w)d(p)$ *is quasi-concave in* $p \in [w, \infty)$ *for any* w. *Then*

$$\begin{aligned} p^*(w) &= \max\{p^c(w), p^m(w)\} \\ &= \begin{cases} p^c(w) \text{ that is decreasing in } w, & \text{for } w \leq \bar{w}, \\ p^m(w) \text{ that is increasing in } w, & \text{for } w > \bar{w}, \end{cases} \end{aligned}$$

where $\bar{w} \equiv \inf\{w \mid p^c(w) < p^m(w)\}$. *(If* $\bar{w}$ *is* ∞, *the latter case is moot.)*

Proof of Theorem 2.1. Since $d(p)$ is decreasing in p and $s(w)$ is increasing in w, $p^c(w)$ is decreasing in w.

Moreover, $p^m(w)$ is increasing in w. This is because, for $p' \geq p$ and $w' \geq w$, $(p' - w')d(p') - (p' - w)d(p') = -(w' - w)d(p') \geq -(w' - w)d(p) = (p - w')d(p) - (p - w)d(p)$, where the inequality is due to that $w' \geq w$ and $d(p)$ is decreasing in p. Rearranging terms in the above inequality, we have $(p' - w')d(p') - (p - w')d(p) \geq (p' - w)d(p') - (p - w)d(p)$, which implies that $(p - w)d(p)$ has increasing differences in (p, w). By Topkis (1998), $p^m(w)$ is increasing in w.

As $w \to 0$, we have $s(w) \to 0$. As mentioned, $p^c(w)$ is decreasing in w. We claim that $p^c(w) \to p^c(0)$ as $w \to 0$. If this does not hold, there would exist $\bar{p}^0 < p^c(0)$ such that $\lim_{w\to 0} p^c(w) = \bar{p}^0 < p^c(0)$. Then, $d(p^c(w)+) \equiv \lim_{p\to p^c(w)+} d(p) \geq d(\bar{p}^0+) > 0$ for all $w > 0$. This contradicts the fact that $\lim_{w\to 0} d(p^c(w)+) \leq \lim_{w\to 0} s(w) = 0$. Thus $p^c(w) \to p^c(0)$ as $w \to 0$.

On the other hand, since $\lim_{p\to p^c(0)} (p-w)d(p) \leq \lim_{p\to p^c(0)} pd(p) = 0$, we have $p^m(w) < p^c(0)$ for all $w > 0$. Moreover, recall that we have shown that $p^c(w)$ is decreasing in w, $p^m(w)$ is increasing in w and $p^c(w) \to p^c(0)$ as $w \to 0$. Then there must exist a sufficiently small $w_0 > 0$ such that $p^c(w_0) > p^m(w_0)$. Let $\bar{w} \equiv \inf \{w \mid p^c(w) < p^m(w)\}$ which can be ∞. We have $\bar{w} > 0$.

Consider the platform's profit maximization problem $\max_p \pi(p; w) = \max_p (p-w)[s(w) \wedge d(p)]$ for a given w. For $p \leq p^c(w)$, $\pi(p; w) = (p-w)s(w)$ is strictly increasing in p provided that $s(w) > 0$. This implies that $p^*(w) \geq p^c(w)$. Therefore,

$$\max_p \pi(p; w) = \max_{p \geq p^c(w)} \pi(p; w) = \max_{p \geq p^c(w)} (p-w)d(p).$$

If $w \leq \bar{w}$, by definition of $\bar{w}$, we have $p^c(w) \geq p^m(w)$. Since $(p-w)d(p)$ is quasi-concave in p, we have $p^*(w) = p^c(w)$, which is decreasing in w $(\leq \bar{w})$.

If $w > \bar{w}$, we have $p^c(w) < p^m(w)$, we have $p^*(w) = p^m(w)$, which is increasing in w $(> \bar{w})$. $\square$

The conditions $\lim_{w\to 0} s(w) = s(0) = 0$ and $\lim_{p\to p^c(0)} pd(p) = 0$ are innocuous. The former says that the available amount of supply diminishes to zero when the wage approaches to zero. It would naturally be satisfied when we derive the supply curve as $s(w) = \mathsf{P}(X \leq w)$, where the supplier opportunity cost X is a continuous random variable. The latter says that the total generated revenue diminishes to zero when the price is sufficiently close to the "choke price" $p^c(0)$. The quasi-concavity condition of $(p-w)d(p)$ in $p \in [w, \infty)$ for any w holds for almost all commonly seen demand functions. In fact, the condition holds for $d(p) = \mathsf{P}(Y \geq p)$, where the distribution of the customer willingness-to-pay Y has an increasing generalized failure rate (see, e.g., Van den Berg 2007).

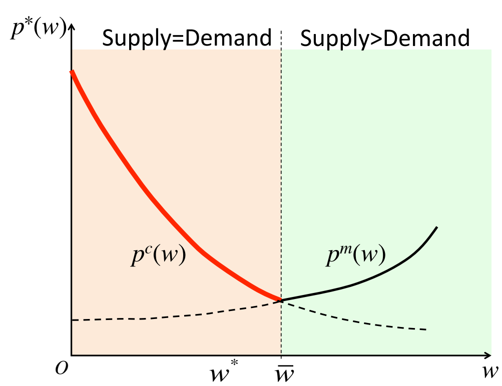


Figure 2.5: Optimal Price as a Function of Exogenous Wage

As the wage increases, there is more supply and the price to clear the supply decreases. Hence the market clearing price $p^c(w)$ is always decreasing in w. As the wage rises, the revenue maximizing price without supply constraints is pushed up, i.e., $p^m(w)$ is always increasing in w. Theorem 2.1 says that for an exogenous wage w, the optimal price is the maximum of the market clearing price $p^c(w)$ and the revenue maximizing price $p^m(w)$. As a result, the platform's optimal price $p^*(w)$ has a *U-shaped* relationship with the wage w (see Figure 2.5 for an illustration). The intuition is as follows. On one hand, when the wage is above a threshold, the supply is ample due to the relatively high wage, and thus the optimal price is the unconstrained revenue maximizing price, which would increase as the wage rises. On the other hand, when the wage is below the threshold, the supply is limited due to the relatively low wage, and thus the optimal price is the market clearing price, i.e., a price such that the sales volume is equal to the supply. As the wage increases in a neighborhood, more supply comes to the market. To match up with the total increased supply, the platform optimally sets a *lower* price! This implies that campaigns to improve wages and benefits such as imposing a higher minimum wage for the independent agents in a two-sided market likely also *benefit* customers on the demand side, when the market tends to be supply constrained due to low wages.

That the optimal price may decrease in wage is in stark contrast with the traditional supply chain settings. Consider a supply chain where a retailer procures from its supplier who may in turn procure from further up-stream suppliers. The retailer faces a downward sloping demand curve $d(p)$ in the consumer market and the supplier charges the retailer a wholesale price w. Any cost surge in the supply chain, e.g., a wage increase at some firm along the supply chain, would push up the wholesale price w and lead to an increase in the optimal retail price $p^*(w) = \arg\max_p (p-w)d(p)$. This is consistent with the monotonicity property of the revenue maximizing price.

That the optimal price may decrease in wage is also in stark contrast to the classic economics literature on two-sided matching. Rochet and Tirole (2003) assume a multiplicative form of transaction volumes. That is, the platform solves the problem $\max_{p,w}(p-w)d(p)s(w)$. It is easy to see that the objective function $(p-w)d(p)s(w)$ here is log-supermodular in p and w. As a results, the optimal price $p^*(w) = \arg\max_p (p-w)d(p)s(w)$ is increasing in w.

Uber often uses "surge pricing" to rapidly match supply with demand. That is, it raises the price when ride demand is higher than the number of drivers available. Our results suggest that unlike conventional views, surge pricing can have a positive effect on riders. Other than rationing the current limited supply to riders who are less patient, price surge and its resulting wage surge can induce more drivers to come out. The platform may have an incentive to lower prices to match the newly arrived drivers. Though our model is one-shot, it may capture the dynamic process in which short-term price surge leads to more supply entering the market, and as a result, the average price over a slightly longer horizon (a weighted sum of the high short-term surge price[3] and the lower price after the surge period

[3] Chen et al. (2015) observe that surge pricing of Uber is often short-lived.

to match the remaining demand with newly arriving supply) is lower.

2.4.2 Joint Price and Wage Control

We will now solve the price and wage optimization problem $\max_{p,w} \pi(p,w) = (p-w)\min\{s(w), d(p)\}$. The following proposition shows that instead of maximizing the profit with respect to p and w, we can first find the optimal matching quantity z^*, from which the optimal price p^* and wage w^* can be recovered.

Theorem 2.2 (MAXIMIZATION OF PLATFORM PROFIT) *Let* $z^* \in \arg\max_{z\geq 0}[d^{-1}(z) - s^{-1}(z)]z$. *Then the optimal price and wage are, respectively,*

$$p^* = d^{-1}(z^*) \quad \textit{and} \quad w^* = s^{-1}(z^*),$$

where $d^{-1}(z) \equiv \max\{p \geq 0 \mid d(p) = z\}$ *and* $s^{-1}(z) \equiv \min\{w \geq 0 \mid s(w) = z\}$.

Proof of Theorem 2.2. Let $m = p-w$ be the profit margin. We can rewrite the profit function $\pi(p,w)$ as $\tilde{\pi}(w,m) = \pi(w+m,w) = m\min\{s(w), d(w+m)\}$. For a given value of m, $\tilde{\pi}(w;m)$ is quasi-concave in w. This is because, $s(w)$ and $d(w+m)$, as monotone functions, are quasi-concave in w, and the quasi-concavity is preserved under minimization. It is easy to see that for a given m, $\tilde{\pi}(w;m)$ achieves the maximum when $s(w) = d(w+m)$. Let $\hat{w}(m) = \min\{w \geq 0 \mid s(w) = d(w+m)\}$ for a given m. Thus $s(\hat{w}(m)) = d(\hat{w}(m)+m)$. We can solve for the largest margin m such that $s(\hat{w}(m)) = d(\hat{w}(m)+m)$ as $m^*(\hat{w}) = d^{-1}(s(\hat{w})) - \hat{w}$. It is clear that $m^*(\hat{w})$ is decreasing in $\hat{w}$ from the monotonicity properties of $d(p)$ and $s(w)$. The optimal price $p^* = m^*(\hat{w}) + \hat{w} = d^{-1}(s(\hat{w}))$. Thus, maximizing the platform's profit can be written as follows,

$$\max_{\hat{w}\geq 0}[d^{-1}(s(\hat{w})) - \hat{w}]s(\hat{w}) = \max_{z\geq 0}[d^{-1}(z) - s^{-1}(z)]z,$$

which is maximized at z^*. As a result, the optimal wage and price can be recovered as $w^* = s^{-1}(z^*)$ and $p^* = d^{-1}(z^*)$. □

Theorem 2.2 reduces the two-dimensional price and wage optimization problem to a one-dimensional problem. The intuition behind is that for a given scenario, in anticipation of the market formation, there is no incentive for the platform to set the price and wage such that there is more supply arriving than the demand or vice versa. That is, it is optimal for the platform to set the price and wage such that the arriving demand is equal to the arriving supply. Hence, the problem reduces to finding the optimal matching quantity, from which the optimal price and wage can be obtained. The optimal matching quantity z^* to $\max_{z\geq 0}[d^{-1}(z) - s^{-1}(z)]z$ becomes easy to obtain when the objective function $[d^{-1}(z) - s^{-1}(z)]z$ is concave, a condition which is satisfied by the following commonly used demand and supply functions.

(D1) $d(p) = (d_0 - \beta p)^\theta$, $d_0, \beta > 0$. In this case, $d^{-1}(z) = (d_0 - z^{1/\theta})/\beta$ and $zd^{-1}(z) = (d_0 z - z^{1+1/\theta})/\beta$, which is concave as long as $\theta > 0$.

(D2) $d(p) = d_0 p^{-\beta\theta}$, $d_0, \beta > 0$. In this case, $d^{-1}(z) = (z/d_0)^{-1/(\beta\theta)}$ and $zd^{-1}(z) = z^{1-1/(\beta\theta)} d_0^{1/(\beta\theta)}$, which is concave as long as $\beta\theta > 1$.

(S1) $s(w) = (s_0 + \alpha w)^\gamma$, $s_0 \geq 0$, $\alpha > 0$. In this case, $zs^{-1}(z) = z(z^{1/\gamma} - s_0)z/\alpha$ is convex in z if $\gamma > 0$.

(S2) $s(w) = s_0 w^{\alpha\gamma}$, $s_0, \alpha > 0$. In this case, $zs^{-1}(z) = z(z/s_0)^{1/(\alpha\gamma)}$ is convex in z if $\gamma > 0$.

Moreover, we can obtain closed-form expressions of z^*, p^* and w^* if supply and demand are linear functions, say, $s(w) = w$ and $d(p) = d_0 - \beta p$. In this case, $d^{-1}(z) = (d_0 - z)/\beta$ and $s^{-1}(z) = z$. Thus, $[d^{-1}(z) - s^{-1}(z)]z = [d_0 - (1+\beta)z]z/\beta$. It is easy to see that $z^* = d_0/[2(1+\beta)]$. It then follows that $p^* = d^{-1}(z^*) = (d_0 - z^*)/\beta = \frac{1+2\beta}{2\beta(1+\beta)} d_0$ and $w^* = s^{-1}(z^*) = z^* = d_0/[2(1+\beta)]$.

2.5 Commission Contracts

Recall that a commission contract f specifies a relation $w = f(p)$ between the price and wage. Given a realized Scenario k (with probability ρ_k), the platform decides on the price p_k, with the wage following from the pre-determined commission contract as $w_k = f(p_k)$, to maximize the profit $\pi_k(p_k, w_k)$ for that scenario. The platform solves the following problem to maximize the expected profit.

$$\begin{aligned} \max_{\{p_k, w_k\}_{\forall k}} \quad & \textstyle\sum_{k \in \mathcal{K}} \rho_k \pi_k(p_k, w_k), \\ \text{s.t.} \quad & w_k = f(p_k). \end{aligned} \tag{2.2}$$

We denote the optimal value of problem (2.2) by P^f.

Let $(p_k^*, w_k^*) \in \arg\max_{p,w} \pi_k(p_k, w_k)$, i.e., p_k^* and w_k^* are the optimal price and wage in Scenario k, respectively, when the platform can freely choose the price and wage. The corresponding optimal expected profit is $P^* \equiv \sum_{k \in \mathcal{K}} \rho_k \pi_k(p_k^*, w_k^*)$. The following result shows that this optimal expected profit can always be almost attained by some commission contract f, if we allow $f : \mathbb{R}_+ \to \mathbb{R}_+$ to take an *arbitrary* functional form.

Proposition 2.2 (Connecting the Dots) *For any $\epsilon > 0$, there exists an ϵ-optimal commission contract f such that $P^* - \epsilon \leq P^f \leq P^*$.*

Proof of Proposition 2.2. Without loss of generality, we assume that $p_1^* \leq p_2^* \leq \cdots \leq p_K^*$.

We choose a sufficiently small number $\delta > 0$ and define $\tilde{p}_k$ recursively. Let $\tilde{p}_1 = p_1^*$, $\tilde{p}_k = \tilde{p}_{k-1} + \delta$ if $\tilde{p}_k = \tilde{p}_{k-1}^*$, and $\tilde{p}_k = p_k^*$ otherwise. By construction, $\lim_{\delta \to 0} \tilde{p}_k = p_k^*$ for all $k \in \mathcal{K}$. For ease of notation, let $\tilde{p}_0 = w_0^* = 0$.

Consider the following piecewise linear function f parameterized by δ. For $p \in [\tilde{p}_{k-1}, \tilde{p}_k]$, let $f(p) = \frac{w_k^* - w_{k-1}^*}{\tilde{p}_k - \tilde{p}_{k-1}}(p - \tilde{p}_{k-1}) + w_{k-1}^*$. Note that the curve $p \to f(p)$ passes all the points $\{(\tilde{p}_k, w_k^*) \mid k = 1, \ldots, K\}$.

In Scenario k, the optimal profit under the commission contract f, namely $\pi_k(f)$, is no less than $\pi_k(\tilde{p}_k, w_k^*)$. Given that d_k is a continuous function, $\lim_{\delta\to 0}\pi_k(f) \geq \lim_{\delta\to 0}\pi_k(\tilde{p}_k, w_k^*) = \pi_k(p_k^*, w_k^*)$. Then, $\lim_{\delta\to 0} P^f = \lim_{\delta\to 0}\sum_{k\in\mathcal{K}}\rho_k\pi_k(f) = \sum_{k\in\mathcal{K}}\rho_k \lim_{\delta\to 0}\pi_k(f) \geq \sum_{k\in\mathcal{K}}\rho_k\pi_k(p_k^*, w_k^*) = P^*$. Thus, for any $\epsilon > 0$, there exists δ with the corresponding function f such that $P^* - \epsilon < P^f \leq P^*$. $\square$

Proposition 2.2 essentially finds a curve $w = f(p)$ that passes through the points $(p_1^*, w_1^*), \ldots, (p_K^*, w_K^*)$, or almost passes through some points $(p_{k'}^*, w_{k'}^*)$ and $(p_{k''}^*, w_{k''}^*)$, if $p_{k'}^* = p_{k''}^*$. The latter perturbation avoids the situation where one price corresponds to multiple wages in the commission contract. Compared with such a commission contract f, the fixed commission contract, defined below, is simpler and more practical.

Definition 2.2 (FIXED COMMISSION CONTRACT) *A fixed commission contract f is a commission contract in the form of $w = f(p) = \gamma p$, where $\gamma \in [0,1]$.*

Under the fixed commission contract, the platform takes the given portion, γ, of the revenue in any scenario. Thus, $1-\gamma$ is the fixed commission rate charged by the platform. Despite its simplicity, the fixed commission contract can achieve the optimal expected profit P^* under certain conditions, as shown by the following proposition.

Proposition 2.3 (WHEN FIXED COMMISSION CONTRACT IS OPTIMAL) *If $d_k(p) = d_{k0} - \beta p$ and $s_k(w) = \alpha w$, then the fixed commission contract $f(p) = \beta(\alpha+2\beta)^{-1}p$ is optimal.*

Proof of Proposition 2.3. If $d_k(p) = d_{k0} - \beta p$ and $s_k(w) = \alpha w$, then $d_k^{-1}(z) = \beta^{-1}(d_{k0} - z)$ and $s_k^{-1}(z) = \alpha^{-1}z$. Then, $[d_k^{-1}(z) - s_k^{-1}(z)]z = [\beta^{-1}(d_{k0} - z) - \alpha^{-1}z]z = -(\beta^{-1}+\alpha^{-1})z^2 + \beta^{-1}d_{k0}z$ attains its maximum at $z_k^* = \frac{1}{2}\beta^{-1}(\alpha^{-1}+\beta^{-1})^{-1}d_{k0} = \frac{1}{2}\alpha(\alpha+\beta)^{-1}d_{k0}$. By Theorem 2.2, the optimal wage and price in Scenario k is given by $w_k^* = s_k^{-1}(z_k^*) = \frac{1}{2}(\alpha+\beta)^{-1}d_{k0}$ and $p_k^* = d_k^{-1}(z_k^*) = \frac{1}{2}\beta^{-1}(\alpha+2\beta)(\alpha+\beta)^{-1}d_{k0}$. Since $w_k^*/p_k^* = \beta(\alpha+2\beta)^{-1}$ is independent of k, the fixed commission contract $f(p) = \beta(\alpha+2\beta)^{-1}p$ is optimal. $\square$

For a fixed commission contract to be optimal, the conditions in Proposition 2.3 require both demand and supply functions to be linear and that only the potential consumer market size d_{k0} varies across different scenarios while the price sensitivities α and β remain the same. If these conditions are not satisfied, we cannot guarantee the optimality of the fixed commission contract. We first present an illustrative numerical example on the performance of the fixed commission contract.

2.5.1 An Illustrative Example of Fixed Commission Contract

Suppose that in a scenario $k \in \mathcal{K}$, the supply function is $s_k(w) = s_{k0}F_{s,k}(w)$ and the demand function is $d_k(p) = d_{k0}[1 - F_{d,k}(p)]$. Here, $F_{s,k}$ (resp. $F_{d,k}$) is the cumulative distribution function (c.d.f.) of the conditional normal distribution (conditioned on nonnegative values) with mean $\mu_{s,k}$ (resp. $\mu_{d,k}$) and standard deviation $\sigma_{s,k}$ (resp. $\sigma_{d,k}$). The supply and demand curves are obtained assuming that a supplier joins the market if and only if the wage exceeds his/her willingness-to-sell (opportunity cost for providing services), a customer joins the market if and only if the price is below his/her willingness-to-pay (valuation of the service), and both supplier's willingness-to-sell and customer's willingness-to-pay follow normal distributions. The parameters s_{k0} and d_{k0} represent the numbers of the potential suppliers and customers, respectively. Recall that ρ_k is the probability for observing Scenario $k \in \mathcal{K}$.

We consider $K = 10$ scenarios. The parameters s_{k0}, d_{k0}, $\mu_{s,k}$, $\sigma_{s,k}$, $\mu_{d,k}$, $\sigma_{d,k}$ and ρ_k are chosen as in Table 2.1. To isolate the effect of the market size, in this example, the pool size of potential suppliers and customers is held fixed ($s_{k0} = 1$, $d_{k0} = 1.2$ for all $k \in \mathcal{K}$). The mean of the suppliers' opportunity cost is increasing across the scenarios and so is the mean of the customers' valuation. Imagine as k increases, the weather condition worsens, and customers value more going by a car more than say, using a bike sharing service, but at the same time, drivers also value staying at home more than driving. To focus on the first order effect, the standard deviation $\sigma_{s,k}$ of the suppliers' cost is set to $\frac{1}{3}$ of the mean $\mu_{s,k}$ and the same is done on the demand side.

In each Scenario k, the optimal price p_k^*, optimal wage w_k^*, and the ratio $\gamma_k^* = w_k^*/p_k^*$ in the benchmark are displayed in Table 2.2. As expected, we see that both price p_k^* and wage w_k^* increase in k. Since both suppliers' cost and customers' valuation increase as the weather condition worsens, the platform needs to increase the wage to attract more suppliers and increase the price to suppress the growing demand. We observe that the optimal wage/price ratio γ_k^* decreases in k. This implies that the platform does not need to increase the commission ratio to incentivize the suppliers. This is because even though γ_k^* decreases as k increases, the optimal wage $w_k^* = \gamma_k^* p_k^*$ is in fact increased and is sufficient for incentivizing the drivers to get on the street.

Table 2.1: Parameters in the Illustrative Example

Scenario k	1	2	3	4	5	6	7	8	9	10
s_{k0}	1	1	1	1	1	1	1	1	1	1
$\mu_{s,k}$	15	16	17	18	19	20	21	22	23	24
$\sigma_{s,k}$	5	5.33	5.67	6	6.33	6.67	7	7.33	7.67	8
d_{k0}	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2
$\mu_{d,k}$	10	12	14	16	18	20	22	24	26	28
$\sigma_{d,k}$	3.33	4	4.67	5.33	6	6.67	7.33	8	8.67	9.33
ρ_k	0.05	0.05	0.1	0.1	0.2	0.2	0.1	0.1	0.05	0.05

Table 2.2: Optimal Wages, Prices and Ratios in the Illustrative Example

Scenario k	1	2	3	4	5	6	7	8	9	10
p_k^*	14.20	16.52	18.79	25.90	27.11	28.32	29.53	30.74	31.95	33.16
w_k^*	9.24	10.59	11.87	13.10	14.28	15.44	16.57	17.69	18.79	19.88
γ_k^*	0.651	0.641	0.631	0.622	0.614	0.606	0.599	0.593	0.587	0.581

When the platform freely sets both wage and price in each scenario, the optimal expected profit of the platform in the benchmark is achieved by applying the wage-price pair (w_k^*, p_k^*) in Scenario k and is equal to $\sum_{k\in\mathcal{K}} \rho_k \pi_k(w_k^*, p_k^*) = 2.311$. Under the fixed commission contract $w = \gamma p$, the optimal expected profit of the platform is achieved by applying the price $\check{p}_k(\gamma) = \arg\max_{p\geq 0} \pi_k(\gamma p, p)$ for given γ in Scenario k. The optimal fixed commission contract for maximizing the platform's profit can be found by solving $\max_{\gamma\in[0,1]} \sum_{k\in\mathcal{K}} \rho_k \pi_k(\gamma\check{p}_k(\gamma), \check{p}_k(\gamma))$. With the given parameters, the optimal ratio $\check{\gamma} = 0.6063$. This fixed commission contract achieves a surprisingly high profit 2.307, which is 99.82% of the optimal profit achieved by $\{(w_k^*, p_k^*)\}_{k\in\mathcal{K}}$.

2.5.2 Performance Bounds of Fixed Commission Contract

Motivated by the illustrative example, we investigate the performance of the fixed commission contract. We will show that under general conditions, the optimal fixed commission contract is indeed able to achieve a decent portion of optimality.

Theorem 2.3 (Concave Supply Curve: 3/4-Optimality of Fixed Commission) *Suppose $s_k(w)$ is concave for all $k \in \mathcal{K}$.*

(i) For any Scenario k, $\gamma_k^* \equiv w_k^*/p_k^* \leq 50\%$.

(ii) The expected profit achieved by a heuristic commission contract with $\gamma = \frac{(1-\bar{\gamma})\bar{\gamma}}{(1-\underline{\gamma})} (\leq 50\%)$ is at least $\frac{3}{4}P^*$, where $\bar{\gamma} = \max_{k\in\mathcal{K}} \gamma_k^*$ and $\underline{\gamma} = \min_{k\in\mathcal{K}} \gamma_k^*$.

Proof of Theorem 2.3. Under a given commission contract $w = \gamma p$ where $\gamma \in [0,1]$, the optimal profit of the platform in Scenario k is $P_k^*(\gamma) = \max_p (1-\gamma)p[s_k(\gamma p) \wedge d_k(p)]$. Define $\gamma_k^* \equiv w_k^*/p_k^*$, where $(p_k^*, w_k^*) \in \arg\max_{p,w} \pi_k(p_k, w_k)$.

(i) If $\gamma \leq \gamma_k^*$, we have $\gamma/\gamma_k^* \leq 1$. By the concavity of $s_k(\cdot)$,

$$s_k(\gamma p) = s_k(\frac{\gamma}{\gamma_k^*} \cdot \gamma_k^* p + (1 - \frac{\gamma}{\gamma_k^*}) \cdot 0) \geq \frac{\gamma}{\gamma_k^*} s_k(\gamma_k^* p) + (1 - \frac{\gamma}{\gamma_k^*}) s_k(0) = \frac{\gamma}{\gamma_k^*} s_k(\gamma_k^* p), \tag{2.3}$$

where the last equality is due to $s_k(0) = 0$ for all k. Then it follows that

$$\begin{aligned} P_k^*(\gamma) =& \max_p (1-\gamma)p[s_k(\gamma p) \wedge d_k(p)] \\ \geq& \max_p (1-\gamma)p\{[\frac{\gamma}{\gamma_k^*} s_k(\gamma_k^* p)] \wedge d_k(p)\} \\ \geq& (1-\gamma)\frac{\gamma}{\gamma_k^*} \max_p p[s_k(\gamma_k^* p) \wedge d_k(p)] \\ =& \frac{(1-\gamma)\gamma}{(1-\gamma_k^*)\gamma_k^*} \max_p (1-\gamma_k^*)p[s_k(\gamma_k^* p) \wedge d_k(p)] \\ =& \frac{(1-\gamma)\gamma}{(1-\gamma_k^*)\gamma_k^*} P_k^*(\gamma_k^*) \\ =& \frac{(1-\gamma)\gamma}{(1-\gamma_k^*)\gamma_k^*} \pi_k(p_k^*, w_k^*), \end{aligned} \tag{2.4}$$

where the first inequality is due to (2.3) and the last equality holds because $\gamma_k^* = w_k^*/p_k^*$, so that $P_k^*(\gamma_k^*)$ equals the optimal profit $\pi_k(p_k^*, w_k^*)$ in Scenario k.

Note that (2.4) holds for an arbitrary $\gamma \leq \gamma_k^*$. Now let $\gamma = \frac{1}{2}$. Suppose $\gamma_k^* > \frac{1}{2}$. Then $(1-\gamma_k^*)\gamma_k^* < \frac{1}{4}$ by the property of the quadratic function $x - x^2$. With $\gamma = \frac{1}{2}$, $[(1-\gamma)\gamma]/[(1-\gamma_k^*)\gamma_k^*] > 1$. Then by (2.4), $P_k^*(\gamma = \frac{1}{2}) > \pi_k(p_k^*, w_k^*)$. However, this contradicts the optimality of $\pi_k(p_k^*, w_k^*)$. Thus, for all k,

$$\gamma_k^* \leq \frac{1}{2},$$

and therefore $\bar{\gamma} \leq \frac{1}{2}$. Then $(1-\gamma_k^*)\gamma_k^*$ is increasing in $\gamma_k^* \in [0, \frac{1}{2}]$. Thus, by (2.4),

$$\frac{P_k^*(\gamma)}{\pi_k(p_k^*, w_k^*)} \geq \frac{(1-\gamma)\gamma}{(1-\gamma_k^*)\gamma_k^*} \geq \frac{(1-\gamma)\gamma}{(1-\bar{\gamma})\bar{\gamma}}. \tag{2.5}$$

(ii) For $\gamma \geq \gamma_k^*$,

$$\begin{aligned} P_k^*(\gamma) =& \max_p (1-\gamma)p[s_k(\gamma p) \wedge d_k(p)] \\ \geq& (1-\gamma)p_k^*[s_k(\gamma p_k^*) \wedge d_k(p_k^*)] \\ \geq& (1-\gamma)p_k^*[s_k(\gamma_k^* p_k^*) \wedge d_k(p_k^*)] \\ =& (1-\gamma)(1-\gamma_k^*)^{-1}(1-\gamma_k^*)p_k^*[s_k(\gamma_k^* p_k^*) \wedge d_k(p_k^*)] \\ =& (1-\gamma)(1-\gamma_k^*)^{-1}\pi_k(p_k^*, w_k^*), \end{aligned}$$

where the second inequality is due to $\gamma \geq \gamma_k^*$. Then it follows that

$$\frac{P_k^*(\gamma)}{\pi_k(p_k^*, w_k^*)} \geq \frac{1-\gamma}{1-\gamma_k^*} \geq \frac{1-\gamma}{1-\underline{\gamma}}. \tag{2.6}$$

By (2.5) and (2.6), for $\gamma \in [\underline{\gamma}, \bar{\gamma}] \subseteq [0, \frac{1}{2}]$,

$$\frac{P_k^*(\gamma)}{\pi_k(p_k^*, w_k^*)} \geq \min\left\{\frac{(1-\gamma)\gamma}{(1-\bar{\gamma})\bar{\gamma}}, \frac{1-\gamma}{1-\underline{\gamma}}\right\} = \frac{1-\gamma}{(1-\bar{\gamma})\bar{\gamma}} \min\left\{\gamma, \frac{(1-\bar{\gamma})\bar{\gamma}}{1-\underline{\gamma}}\right\} = \begin{cases} \dfrac{(1-\gamma)\gamma}{(1-\bar{\gamma})\bar{\gamma}} & \text{if } \gamma \leq \dfrac{(1-\bar{\gamma})\bar{\gamma}}{1-\underline{\gamma}}, \\ \dfrac{1-\gamma}{1-\underline{\gamma}} & \text{otherwise.} \end{cases}$$

The right-hand-side of the above inequality achieves the maximum value, which is equal to $(1-\underline{\gamma})^{-1}\left[1-(1-\underline{\gamma})^{-1}(1-\bar{\gamma})\bar{\gamma}\right]$, when

$$\gamma = \gamma^* \equiv \frac{(1-\bar{\gamma})\bar{\gamma}}{(1-\underline{\gamma})}.$$

Let $y = (1-\underline{\gamma})^{-1}$. Note that $\underline{\gamma}$ can range from 0 to $\bar{\gamma}$. Thus, y ranges from 1 to $(1-\bar{\gamma})^{-1}$. Then, for any $k \in \mathcal{K}$, we have

$$\frac{P_k^*(\gamma^*)}{\pi_k(p_k^*, w_k^*)} \geq y[1 - y(1-\bar{\gamma})\bar{\gamma}].$$

The right-hand-side $y[1-y(1-\bar{\gamma})\bar{\gamma}]$ is increasing for $0 \leq y \leq [2(1-\bar{\gamma})\bar{\gamma}]^{-1}$ and decreasing for $y > [2(1-\bar{\gamma})\bar{\gamma}]^{-1}$. Given that $\bar{\gamma} \leq \frac{1}{2}$, we have $[2(1-\bar{\gamma})\bar{\gamma}]^{-1} \geq (1-\bar{\gamma})^{-1}$. Therefore, the range of y, $[1, (1-\bar{\gamma})^{-1}]$, is to the left of the changeover point $[2(1-\bar{\gamma})\bar{\gamma}]^{-1}$, implying that the minimum of $y[1-y(1-\bar{\gamma})\bar{\gamma}]$ with respect to y is achieved at $y=1$. When $y=1$, $y[1-y(1-\bar{\gamma})\bar{\gamma}] = 1-(1-\bar{\gamma})\bar{\gamma} \geq \frac{3}{4}$ since $\bar{\gamma} \in [0, \frac{1}{2}]$. As a result, for any $k \in \mathcal{K}$,

$$\frac{P_k^*(\gamma^*)}{\pi_k(p_k^*, w_k^*)} \geq \min_{y \in [1,(1-\bar{\gamma})^{-1}]} y[1-y(1-\bar{\gamma})\bar{\gamma}] = 1-(1-\bar{\gamma})\bar{\gamma} \geq \frac{3}{4}.$$

It follows that

$$\frac{\sum_{k\in\mathcal{K}} \rho_k P_k^*(\gamma^*)}{\sum_{k\in\mathcal{K}} \rho_k \pi_k(p_k^*, w_k^*)} \geq \frac{\sum_{k\in\mathcal{K}} \rho_k \cdot \frac{3}{4} \cdot \pi_k(p_k^*, w_k^*)}{\sum_{k\in\mathcal{K}} \rho_k \pi_k(p_k^*, w_k^*)} = \frac{3}{4},$$

which proves the proposition. □

Surprisingly, Theorem 2.3 imposes no assumption on the demand function $d_k(p)$, except that it should be a downward-sloping curve. This is due to that the result in Theorem 2.3(i) only depends on the concavity of the supply curve. Note that in the benchmark, the platform sets price and wage such that the demand is equal to the supply for any given market condition. As a result, the upper bound on the optimal wage and price ratio may only depend on the shape of the curve of one side, say, the supply side, but not on that of the other side. Theorem 2.3(ii) says that as long as the supply curve is concave, a fixed commission contract can achieve 75% of the optimality of the benchmark. This constant bound on the performance is obtained based on Theorem 2.3(i). The idea is that though for a given realized scenario, the platform may not be able to choose the optimal wage and price ratio γ_k^* in the benchmark, by carefully choosing a fixed commission rate, the optimality loss is not too large. The heuristic commission rate is obtained by maximizing the lower bound on the fraction of optimality achieved with respect to the commission rate.

Theorem 2.3(i) says that for any scenario the ratio of the optimal wage to price in the benchmark should be no more than 50% (i.e., the commission rate is over 50%). This ratio may seem overly low and maybe caused by lack of competition in the model. Uber currently leaves the drivers a fraction of 70%-80% of fares paid by the riders. Thus, Theorem 2.3(i) may imply that the Uber's current pricing practice is not profit maximizing, which is consistent with the news reports saying that Uber is not making profit. Indeed, in practice, the platform may charge a lower commission due to fairness concerns and to improve supplier welfare. Nevertheless, platforms like Uber have strong bargaining powers over their suppliers and may try to raise the commission to increase profit if it would not overly irritating the suppliers. For example, Uber and Lyft started with a 20% commission, and later increased the rate to 25% in most cities. Currently, the Lyft fee in New York City is 31.4%. Bringing the commission below 50% would accordingly lower the performance bound (see, e.g., Theorem 2.4 with possibly higher wage/price ratio and lower performance bound, where the supply curve can be nonconcave). Nevertheless, a lower performance bound does not necessarily imply significantly poorer performance of the fixed commission contract. As we will show numerically, even if the supply function is nonconcave (in which case a performance ratio lower than 75% can be guaranteed), the fixed commission contract still achieves good performance.

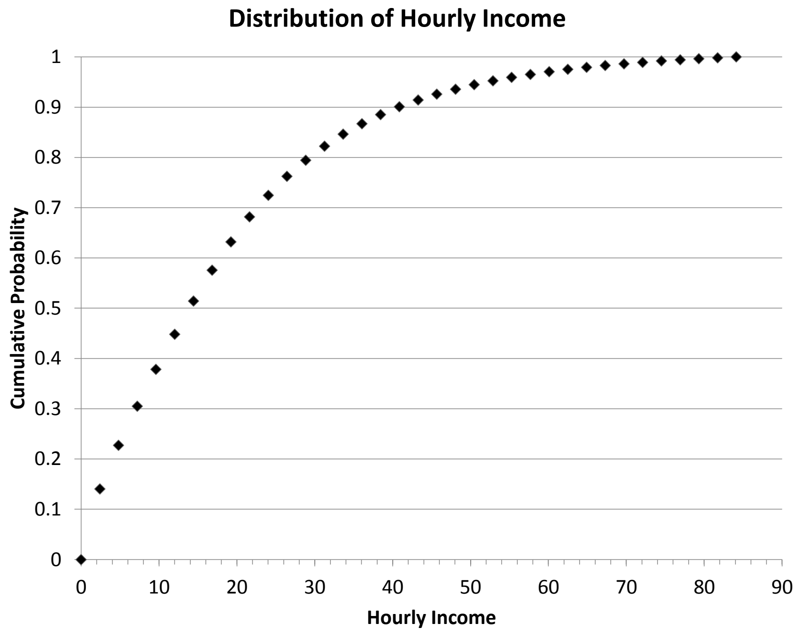


Figure 2.6: Net Hourly Earning Distribution of US in 2015

To obtain some ideas on the shape of the supply function $s_k(w)$, we consider the latest dataset from Social Security Administration on the distribution of wage earners in the United States, which is for the year of 2015.[4] The dataset counts the numbers of wage earners in different annual net compensation intervals. We focus on those with annual net compensation below \$180,000 (consistent with our focal group of Uber drivers), who are grouped into net compensation intervals of of length \$5,000. For example, the dataset records a total number of 10,963,340 wage earners with annual net earnings between \$20,000 and \$24,999.99. We assume that a typical wage earner works 40 hours a week, 52 weeks. Then, the net annual compensation intervals can be converted to net hourly compensation intervals. For example, the interval $[\$20,000, \$24,999.99]$ is converted to $[\$9.62, \$12.02]$. For each net hourly compensation interval, we use the upperbound of the interval (e.g., \$12.02) as the proxy of net hourly earning for all wage earners within that interval, and plot the cumulative distribution function of hourly earning in Figure 2.6. The distribution of net hourly earning can serve as a proxy for the distribution of willingness-to-sell/opportunity cost of the independent suppliers of a sharing economy platform such as Uber, which is different from the supply function up to a scalar. Figure 2.6 shows that the distribution of opportunity cost in 2015 is concave.

If the supply function is not concave, we study the performance of the fixed commission contract under the following assumption.

[4]The data is available at https://www.ssa.gov/cgi-bin/netcomp.cgi?year=2015.

ASSUMPTION (S). $s_k(y)/s_k(x) \geq (y/x)^\lambda$ for all $y \leq x$ and $k \in \mathcal{K}$.

Assumption (S) is equivalent to the following condition: $[\log s_k(x) - \log s_k(y)]/(\log x - \log y) \leq \lambda$ for all $y \leq x$ and $k \in \mathcal{K}$. In other words, the increasing rate of the supply function with respect to the wage is bounded from above by λ in a log-log plot. If $s_k(0) = 0$ and $s_k(w)$ is concave in w, Assumption (S) holds with $\lambda = 1$.

Theorem 2.4 (SUPPLY CURVE WITH BOUNDED GROWTH: BOUND OF FIXED COMMISSION) *Suppose Assumption (S) holds.*

(i) For any Scenario k, $\gamma_k^* \equiv w_k^*/p_k^* \leq \lambda(1+\lambda)^{-1}$.

(ii) The expected profit achieved by a heuristic commission contract with $\gamma = [(1-\bar{\gamma})/(1-\underline{\gamma})]^{\lambda^{-1}}\bar{\gamma}$ is at least a fraction, $1-(1+\lambda)^{-(1+\lambda^{-1})}\lambda$, of the optimal expected profit P^*.

Proof of Theorem 2.4. Suppose that $s_k(y)/s_k(x) \geq (y/x)^\lambda$ for all $k \in \mathcal{K}$ and $y \leq x$. Then, for $\gamma \leq \gamma_k^*$, we have $s_k(\gamma p)/s_k(\gamma_k^* p) \geq (\gamma/\gamma_k^*)^\lambda$. It follows that

$$\begin{aligned}
P_k^*(\gamma) = \max_p (1-\gamma)[s_k(\gamma p) \wedge d_k(p)] \geq &(1-\gamma)\max_p \{[(\gamma/\gamma_k^*)^\lambda s_k(\gamma_k^* p)] \wedge d_k(p)\} \\
\geq &(1-\gamma)(\gamma/\gamma_k^*)^\lambda \max_p \{s_k(\gamma_k^* p) \wedge d_k(p)\} \\
= &\frac{(1-\gamma)\gamma^\lambda}{(1-\gamma_k^*)(\gamma_k^*)^\lambda}\pi_k(p_k^*, w_k^*),
\end{aligned}$$

which implies that

$$P_k^*(\gamma)/\pi_k(p_k^*, w_k^*) \geq (1-\gamma)\gamma^\lambda/[(1-\gamma_k^*)(\gamma_k^*)^\lambda]. \tag{2.7}$$

It is easy to verify that

$$(1-\gamma)\gamma^\lambda \leq (1+\lambda)^{-(1+\lambda)}\lambda^\lambda, \tag{2.8}$$

with the equality achieved at $\gamma = \lambda(1+\lambda)^{-1}$.

Next, we show that $\gamma_k^* \leq \lambda(1+\lambda)^{-1}$. If otherwise, we can set $\gamma = \lambda(1+\lambda)^{-1}$. Then, we have $\max_p \pi_k(p, \gamma p)/\max_p \pi_k(p, \gamma_k^* p) \geq (1-\gamma)\gamma^\lambda/[(1-\gamma_k^*)(\gamma_k^*)^\lambda] > 1$, which contradicts the optimality of γ_k^*. Furthermore, since $(1-\gamma)\gamma^\lambda$ is increasing in γ for $\gamma \in [0, \lambda(1+\lambda)^{-1}]$, $(1-\gamma_k^*)(\gamma_k^*)^\lambda \leq (1-\bar{\gamma})\bar{\gamma}^\lambda$. Then by (2.7), we have $P_k^*(\gamma)/\pi_k(p_k^*, w_k^*) \geq (1-\gamma)\gamma^\lambda/[(1-\bar{\gamma})\bar{\gamma}^\lambda]$ for $\gamma \leq \gamma_k^*$.

For $\gamma > \gamma_k^*$, we have $P_k^*(\gamma)/\pi_k(p_k^*, w_k^*) = (1-\gamma)\max_p p[s_k(\gamma p) \wedge d_k(p)]/\{(1-\gamma_k^*)\max_p p[s_k(\gamma_k^* p) \wedge d_k(p)]\} \geq (1-\gamma)/(1-\gamma_k^*) \geq (1-\gamma)/(1-\underline{\gamma})$, where the first inequality is due to $\gamma > \gamma_k^*$ and the second inequality is due to $\gamma_k^* \geq \underline{\gamma}$.

For $\gamma \in [0,1]$ and any $k \in \mathcal{K}$, we have

$$P_k^*(\gamma)/\pi_k(p_k^*, w_k^*) \geq \min\left\{\frac{(1-\gamma)\gamma^\lambda}{(1-\bar{\gamma})\bar{\gamma}^\lambda}, \frac{1-\gamma}{1-\underline{\gamma}}\right\} = \frac{(1-\gamma)}{(1-\bar{\gamma})\bar{\gamma}^\lambda}\min\left\{\gamma^\lambda, \frac{(1-\bar{\gamma})\bar{\gamma}^\lambda}{1-\underline{\gamma}}\right\}$$

$$= \begin{cases} \dfrac{(1-\gamma)\gamma^{\lambda}}{(1-\bar{\gamma})\bar{\gamma}^{\lambda}} & \text{if } \gamma \le \left(\dfrac{1-\bar{\gamma}}{1-\underline{\gamma}}\right)^{\lambda^{-1}} \bar{\gamma}, \\ \dfrac{1-\gamma}{1-\underline{\gamma}} & \text{otherwise.} \end{cases} \tag{2.9}$$

Note that $[(1-\bar{\gamma})/(1-\underline{\gamma})]^{\lambda^{-1}}\bar{\gamma} \le \bar{\gamma} \le \lambda(1+\lambda)^{-1}$. Since $(1-\gamma)\gamma^{\lambda}$ is increasing in γ for $\gamma \le \lambda(1+\lambda)^{-1}$, it is increasing in γ for $\gamma \le [(1-\bar{\gamma})/(1-\underline{\gamma})]^{\lambda^{-1}}\bar{\gamma}$. The right-hand-side of (2.9) is maximized at

$$\gamma^* = [(1-\bar{\gamma})/(1-\underline{\gamma})]^{\lambda^{-1}}\bar{\gamma},$$

with the maximum value equal to $(1-\underline{\gamma})^{-1}\left\{1-[(1-\bar{\gamma})/(1-\underline{\gamma})]^{\lambda^{-1}}\bar{\gamma}\right\}$.

Let $y = (1-\underline{\gamma})^{-1}$. Then $y \in [1, (1-\bar{\gamma})^{-1}]$ and

$$(1-\underline{\gamma})^{-1}\left\{1-[(1-\bar{\gamma})/(1-\underline{\gamma})]^{\lambda^{-1}}\bar{\gamma}\right\} = y\left\{1-(1-\bar{\gamma})^{\lambda^{-1}}\bar{\gamma}y^{\lambda^{-1}}\right\}.$$

Note that the right-hand-side of the above equation, $y\left\{1-(1-\bar{\gamma})^{\lambda^{-1}}\bar{\gamma}y^{\lambda^{-1}}\right\}$, is increasing in y for $y \le \left[(1-\bar{\gamma})\bar{\gamma}^{\lambda}(1+\lambda^{-1})^{\lambda}\right]^{-1}$, and decreasing in y for $y \ge \left[(1-\bar{\gamma})\bar{\gamma}^{\lambda}(1+\lambda^{-1})^{\lambda}\right]^{-1}$. Since $\gamma_k^* \le \lambda(1+\lambda)^{-1}$, $\bar{\gamma} \le \lambda(1+\lambda)^{-1} = (1+\lambda^{-1})^{-1}$. Hence, $(1-\bar{\gamma})^{-1} \le \left[(1-\bar{\gamma})\bar{\gamma}^{\lambda}(1+\lambda^{-1})^{\lambda}\right]^{-1}$. Then the function $y\left\{1-(1-\bar{\gamma})^{\lambda^{-1}}\bar{\gamma}y^{\lambda^{-1}}\right\}$ is increasing in y over $y \in [1, (1-\bar{\gamma})^{-1}]$ and thus achieves the minimum at $y = 1$. As a result, $(1-\underline{\gamma})^{-1}\left\{1-[(1-\bar{\gamma})/(1-\underline{\gamma})]^{\lambda^{-1}}\bar{\gamma}\right\} \ge 1-(1-\bar{\gamma})^{\lambda^{-1}}\bar{\gamma} = 1-[(1-\bar{\gamma})\bar{\gamma}^{\lambda}]^{\lambda^{-1}} \ge 1-[(1+\lambda)^{-(1+\lambda)}\lambda^{\lambda}]^{\lambda^{-1}} = 1-[(1+\lambda)^{-(1+\lambda^{-1})}\lambda]$, where the second inequality is due to (2.8). Thus, we have shown that for any $k \in \mathcal{K}$, $P_k^*(\gamma^*)/\pi_k(p_k^*, w_k^*) \ge 1-(1+\lambda)^{-(1+\lambda^{-1})}\lambda$ and hence, $\sum_{k\in\mathcal{K}}\rho_k P_k^*(\gamma^*)/\sum_{k\in\mathcal{K}}\rho_k\pi_k(p_k^*, w_k^*) \ge 1-(1+\lambda)^{-(1+\lambda^{-1})}\lambda$. □

Proposition 2.4 shows for a given λ that bounds the growth rate of the supply curve in the log-log plot, the upper bound $\lambda(1+\lambda)^{-1}$ on optimal commission ratio and the relative-to-optimal ratio $1-(1+\lambda)^{-(1+\lambda^{-1})}\lambda$. The former increases and the latter decreases in λ. See Table 2.3 for computed ratios corresponding some values of λ. We can see that if Uber's 80% commission rate was indeed optimal, the supply curve would grow as fast as the quartic function. We can also see that if the supply curve increases slower than the cubic function, the worst case ratio on the performance of the fixed commission contract, compared to the optimal contingent pricing, is over 52%. Note that the percentages in Table 2.3 are just guaranteed performance lower bounds. The actual performance of the fixed commission contract, as we will show in Section 2.5.3, is often much better.

Table 2.3: Guaranteed Performance Bound of Fixed Commission Contract

λ	1	2	3	4
Upper bound on optimal commission ratio	50%	66.67%	75%	80%
Guaranteed relative-to-optimal ratio	75%	61.51%	52.75%	46.50%

2.5.3 Numerical Study

We further numerically investigate the performance of the fixed commission contract with randomly generated instances. As in the illustrative example, we consider conditional normal distribution for supplier's opportunity costs and customers' valuations. For the conditional-normally distributed supply cost, the supply curve is neither convex nor concave. We consider $K = 48$ scenarios in total. Following the same notation as in Subsection 2.5.1, we draw the parameters s_{k0}, $\mu_{s,k}$, $\sigma_{s,k}$, d_{k0}, $\mu_{d,k}$ and $\sigma_{d,k}$ independently and uniformly at random, in the following manner: $s_{k0} \sim \mathcal{U}[0,1]$, $\mu_{s,k} \sim \mathcal{U}[10,20]$, $\sigma_{s,k} \sim \mathcal{U}[0.1\mu_{s,k}, 0.4\mu_{s,k}]$, $d_{k0} \sim \mathcal{U}[0,1]$, $\mu_{d,k} \sim \mathcal{U}[10,20]$ and $\sigma_{s,k} \sim \mathcal{U}[0.1\mu_{s,k}, 0.4\mu_{s,k}]$, where $\mathcal{U}[A,B]$ denotes the uniform distribution over the interval $[A,B]$. To generate the probabilities ρ_k, $k \in \mathcal{K}$, we first draw $\tilde{\rho}_k \sim \mathcal{U}[0,1]$ for every k, and then normalize the $\tilde{\rho}_k$'s, i.e., $\rho_k = \tilde{\rho}_k / \sum_{k\in\mathcal{K}} \tilde{\rho}_k$.

We generate a total number of 400 instances of a combination of parameters for the $K = 48$ number of scenarios, and summarize the statistics on the performance of the best fixed commission contract in Table 2.4. The results show that the performance of the best fixed commission contract is consistently good, with the worst case achieving 82.54% of the optimality.

Table 2.4: Statistics of the Performance of the Fixed Commission Contract: Normal Distributions

Maximum	Minimum	Mean	Median	Standard Deviation
96.30%	82.54%	91.07%	91.33%	2.32%

We further generate 400 instances where the suppliers' cost X and customers' valuation Y follow log-normal distributions, i.e., $\log(X)$ follows a normal distribution with mean $\mu_{s,k}$ and standard deviation $\sigma_{s,k}$, and $\log(Y)$ follows a normal distribution with mean $\mu_{d,k}$ and standard deviation $\sigma_{d,k}$. The number of scenarios is still $K = 48$. We randomly draw all the parameters in the same way as we did for the normal distributions. Table 2.5 shows the performance of the fixed commission contract. While the performance is slightly worse compared with that under the normal distributions, it is still consistently good, with even the worst case performing better than the performance guarantee 75% we obtained for the case of concave supply curves.

Table 2.5: Statistics of the Performance of the Fixed Commission Contract: Log-Normal Distributions

Maximum	Minimum	Mean	Median	Standard Deviation
95.43%	76.52%	88.35%	88.59%	3.23%

2.6 Extensions

2.6.1 Maximizing Platform Profit plus Supplier Surplus, or Social Welfare

In the benchmark model of Section 2.3, the platform contingently set price and wage to maximize profit. Here we consider other objectives of the platform. We concentrate on one scenario $k \in \mathcal{K}$, and thus drop the index k in all relevant functions.

To model the supply side and demand side surplus, let s_0 and d_0 be the potential amount of supply and demand in the market, respectively. Suppose that X is the opportunity cost of a randomly selected supplier and Y is the valuation of a randomly selected customer. Only the suppliers with opportunity cost lower than the wage w and customers with valuation higher than the price p will enter the market. Then, the supply curve and the demand curve are given as, $s(w) = s_0\mathsf{P}(X \leq w) = s_0 F_X(w)$ and $d(p) = d_0\mathsf{P}(Y \geq p) = d_0[1 - F_Y(p)]$, where F_X and F_Y are the c.d.f.s of continuous random variables X and Y, respectively. We will use $\bar{F}_X$ and $\bar{F}_Y$ to denote the complements of F_X and F_Y, respectively.

For any unit of matched supply with an opportunity cost c, the supplier earns a surplus $w - c$. Let $f_{X|X\leq w}$ denote the probability density function (p.d.f.) of X conditioned on the event $\{X \leq w\}$, and $f_{Y|Y\geq p}$ denote the p.d.f. of Y conditioned $\{Y \geq p\}$. The likelihood of a matched supply unit having opportunity cost c is $f_{X|X\leq w}(c) = f_X(c)/F_X(w)$. Similarly, the likelihood of a matched demand unit having valuation v is $f_{Y|Y\geq p}(v) = f_Y(v)/[1 - F_Y(p)]$. Thus the total surplus on the supply side is

$$U^s(p, w) = [s(w) \wedge d(p)] \int_0^w \frac{(w-c)f_X(c)}{F_X(w)} dc = [s(w) \wedge d(p)] \frac{\int_0^w F_X(c)dc}{F_X(w)},$$

where the last equation is due to integration by parts. Similarly, the total surplus on the demand side is

$$U^d(p, w) = [s(w) \wedge d(p)] \int_p^{\bar{v}} \frac{(v-p)f_Y(v)}{1 - F_Y(p)} dv = [s(w) \wedge d(p)] \frac{\int_p^{\bar{v}} \bar{F}_Y(v)dv}{\bar{F}_Y(p)}.$$

For on-demand matching platforms, it is often important to maintain good relationship with their independent suppliers. For example, Uber call their drivers "partners." Moreover, like revenue-sharing contracts, the commission contract tends to align the platform's incentive with that of the drivers plus the platform. Thus, in the following we consider the platform's incentive as maximizing the total surplus on the supply side plus the platform's profit, i.e.,

$$\begin{aligned}\max_{p,w} \quad U(p, w) \equiv & U^s(p, w) + (p - w)[s(w) \wedge d(p)] \\ = & [s(w) \wedge d(p)] \left[p - w + \frac{\int_0^w F_X(c)dc}{F_X(w)}\right] = [s(w) \wedge d(p)] \left[p - w + \frac{\int_0^w s(c)dc}{s(w)}\right].\end{aligned}$$

Let $(\tilde{p}, \tilde{w})$ be the optimal solution to the above problem. The following result characterizes $(\tilde{p}, \tilde{w})$ by reducing the problem to a one-dimensional optimization problem, analogous to Theorem 2.2.

Proposition 2.4 (Maximization of Supplier Surplus and Platform Profit) *The optimal price $\tilde{p}$ and wage $\tilde{w}$ that maximize the supply side surplus plus the platform's profit are given by $\tilde{p} = d^{-1}(\tilde{z})$ and $\tilde{w} = s^{-1}(\tilde{z})$, respectively, where $\tilde{z}$ is the optimal solution to $\max_{z\geq 0}\left\{z[d^{-1}(z) - s^{-1}(z)] + \int_0^{s^{-1}(z)} s(c)dc\right\}$, and $d^{-1}(z)$, $s^{-1}(z)$ are defined in Theorem 2.2.*

Proof of Proposition 2.4. For simplicity of exposure, we assume without loss of generality that the functions are differentiable in all subsequent proofs. If not, we could replace derivatives with differences.

To maximize $U(p,w)$, note that the function $h(w) \equiv w - [\int_0^w s(c)dc]/s(w)$ is increasing in w. To see this, $h'(w) = 1 - [s(w)^2 - s'(w)\int_0^w s(c)dc]/s(w)^2 = s'(w)\int_0^w s(c)dc/s(w)^2 \geq 0$. Substitute p with $p = h(w) + m$, then $U(p,w) = U(h(w)+m, w) = m[s(w) \wedge d(h(w)+m)]$. Given $m \geq 0$, the optimal w, denoted by $\tilde{w}(m)$, should equate $s(w)$ and $d(h(w)+m)$. Let $s(\tilde{w}(m)) = d(h(\tilde{w}(m)) + m) = z$. Then, $\tilde{w}(m) = s^{-1}(z)$ and $m = d^{-1}(z) - h(\tilde{w}(m)) = d^{-1}(z) - h(s^{-1}(z))$. As m varies from 0 to ∞, z ranges from $\bar{z}$ to 0, where $\bar{z}$ is the solution to $d^{-1}(z) = h(s^{-1}(z))$. Note that this range of z coincides with the range of z that keeps $m = d^{-1}(z) - h(s^{-1}(z))$ nonnegative. The maximization of $U(p,w)$ can then be rewritten as $\max_{z\geq 0} z[d^{-1}(z) - h(s^{-1}(z))] = \max_{z\geq 0}\left\{z[d^{-1}(z) - s^{-1}(z)] + \int_0^{s^{-1}(z)} s(c)dc\right\}$. If $\tilde{z} \in \arg\max_{z\geq 0} z[d^{-1}(z) - h(s^{-1}(z))]$, we have $\tilde{w} = s^{-1}(\tilde{z})$ and $\tilde{p} = d^{-1}(\tilde{z})$. □

To incentivize a self-interested platform to use the price $\tilde{p}$ and wage $\tilde{w}$, a side payment from the suppliers to the platform plus a wage-price schedule can be used to coordinate the suppliers and the platform.

Now we consider maximizing the aggregate social welfare $W(p,w)$, which is the sum of the platform's profit and the total surplus of both sides of the market:

$$\begin{aligned}\max_{p,w} \quad W(p,w) \equiv& (p-w)[s(w)\wedge d(p)] + U^s(p,w) + U^d(p,w)\\ =&[s(w)\wedge d(p)]\left[p - w + \frac{\int_0^w F_X(c)dc}{F_X(w)} + \frac{\int_p^{\bar{v}} \bar{F}_Y(v)dv}{\bar{F}_Y(p)}\right]\\ =&[s(w)\wedge d(p)]\left[p - w + \frac{\int_0^w s(c)dc}{s(w)} + \frac{\int_p^{\bar{v}} d(v)dv}{d(p)}\right].\end{aligned}$$

Let $(\hat{p}, \hat{w})$ be the optimal solution to the above problem. Like before, the problem can be reduced to a one-dimensional optimization problem.

Proposition 2.5 (Maximization of Social Welfare) *The optimal price $\hat{p}$ and wage $\hat{w}$ that maximize the social welfare are given by $\hat{p} = d^{-1}(\hat{z})$ and $\hat{w} = s^{-1}(\hat{z})$, respectively, where $\hat{z}$ is the optimal solution to $\max_z\left\{z[d^{-1}(z) - s^{-1}(z)] + \int_{d^{-1}(z)}^{\bar{v}} d(v)dv + \int_0^{s^{-1}(z)} s(c)dc\right\}$, and $d^{-1}(z)$, $s^{-1}(z)$ are defined in Theorem 2.2.*

Proof of Proposition 2.5. To maximize $W(p,w)$, note that $g(p) = p + \int_p^{\bar{v}} d(v)dv/d(p)$ is increasing in p because $g'(p) = -d'(p)\int_p^{\bar{v}} d(v)dv/d(p)^2 \geq 0$. Then, we know that $g(p)$ ranges from $g(0) = \int_0^{\bar{v}} d(v)dv/d(0)$ to ∞. Since $\lim_{w\to 0} s(w)/s'(w) = 0$, we have $h(0) = 0$. Since $h(w) \in [0, h(\infty)]$, $\ell \equiv g(p) - h(w) \in [g(0) - h(\infty), \infty)$ and $p = g^{-1}(\ell + h(w))$. For a given $\ell \geq 0$, the social welfare becomes $W(p,w) =$

$\ell[d(g^{-1}(\ell+h(w)))\wedge s(w)]$. If $d(g^{-1}(\ell+h(\infty)))>s(\infty)$, then $d(g^{-1}(\ell+h(w)))>s(w)$ for all w, because $d(g^{-1}(\ell+h(w)))-s(w)$ is decreasing in w. Thus $W(p,w)=\ell s(w)$, which can be further improved if we increase ℓ. As a result, to achieve the optimality, we consider ℓ such that $d(g^{-1}(\ell+h(\infty)))\leq s(\infty)$, which implies the existence of $\hat{w}(\ell)$ such that $d(g^{-1}(\ell+h(\hat{w}(\ell))))=s(\hat{w}(\ell))$. Clearly, $\hat{w}(\ell)$ maximizes the social welfare for given ℓ. Let $z=d(g^{-1}(\ell+h(\hat{w}(\ell))))=s(\hat{w}(\ell))$ and we get $\hat{w}(\ell)=s^{-1}(z)$ and $\ell=g(d^{-1}(z))-h(s^{-1}(z))=d^{-1}(z)-s^{-1}(z)+z^{-1}[\int_{d^{-1}(z)}^{\bar{v}}d(v)dv+\int_0^{s^{-1}(z)}s(c)dc]$. Thus, to maximize the social welfare $W(p,w)$, we first solve $\max_z\left\{z[d^{-1}(z)-s^{-1}(z)]+\int_{d^{-1}(z)}^{\bar{v}}d(v)dv+\int_0^{s^{-1}(z)}s(c)dc\right\}$. Denote the optimal solution by $\hat{z}$. Then, $\hat{w}=s^{-1}(\hat{z})$ and $\hat{p}=d^{-1}(\hat{z})$. □

Analogously, a coordination contract in the form of side payments from both suppliers and customers to the platform plus a price-wage schedule can be used to induce a self-interested platform to achieve the optimal social welfare.

Having characterized the optimal solutions to the three maximization problems, we are now ready to compare the optimal prices, wages and various performance measures under different objectives.

Theorem 2.5 (Comparison) *Compare the three problems with different objectives: the problem of maximizing the platform's profit (denoted by *), maximizing the platform's profit and supplier surplus (denoted by ˜) and maximizing the social welfare (denoted by ˆ):*

(a) (Price) $p^*\geq\tilde{p}\geq\hat{p}$.

(b) (Wage) $w^*\leq\tilde{w}\leq\hat{w}$.

(c) (Volume) $z^*\leq\tilde{z}\leq\hat{z}$.

(d) (Platform's Profit) $\pi(p^*,w^*)\geq\pi(\tilde{p},\tilde{w})\geq\pi(\hat{p},\hat{w})$.

(e) (Demand Surplus) $U^d(p^*,w^*)\leq U^d(\tilde{p},\tilde{w})\leq U^d(\hat{p},\hat{w})$.

(f) (Supplier Surplus) $U^s(p^*,w^*)\leq U^s(\tilde{p},\tilde{w})\leq U^s(\hat{p},\hat{w})$.

(g) (Social Welfare) $W(p^*,w^*)\leq W(\tilde{p},\tilde{w})\leq W(\hat{p},\hat{w})$.

Proof of Theorem 2.5. (a)-(c) The results follow from Theorem 2.2 and Proposition 2.5. In particular, since $(d/dz)\left[\int_0^{s^{-1}(z)}s(c)dc\right]=[s^{-1}(z)]'z\geq0$, we have $\tilde{z}\geq z^*$, which implies that $\tilde{w}=s^{-1}(\tilde{z})\geq s^{-1}(z^*)=w^*$ and $\tilde{p}=d^{-1}(\tilde{z})\leq d^{-1}(z^*)=p^*$.

Because

$$(d/dz)\left[\int_{d^{-1}(z)}^{\bar{v}}d(v)dv+\int_0^{s^{-1}(z)}s(c)dc\right]=-z[d^{-1}(z)]'+z[s^{-1}(z)]'\geq z[s^{-1}(z)]'$$
$$=(d/dz)\left[\int_0^{s^{-1}(z)}s(c)dc\right],$$

where the inequality is due to that $d^{-1}(z)$ is a decreasing function, we have $\hat{z}\geq\tilde{z}$. This implies that $\hat{p}\leq\tilde{p}$ and $\hat{w}\geq\tilde{w}$.

(d) $\hat{z}\left[d^{-1}(\hat{z})-s^{-1}(\hat{z})\right]+\int_0^{s^{-1}(\hat{z})}s(c)dc=U(\hat{p},\hat{w})\leq U(\tilde{p},\tilde{w})=\tilde{z}\left[d^{-1}(\tilde{z})-s^{-1}(\tilde{z})\right]+\int_0^{s^{-1}(\tilde{z})}s(c)dc$, where the inequality is due to the fact that $\tilde{z}$ maximizes the platform's profit plus supplier surplus. It follows that $\pi(\hat{p},\hat{w})=\hat{z}\left[d^{-1}(\hat{z})-s^{-1}(\hat{z})\right]\leq\tilde{z}\left[d^{-1}(\tilde{z})-s^{-1}(\tilde{z})\right]+\int_0^{s^{-1}(\tilde{z})}s(c)dc-\int_0^{s^{-1}(\hat{z})}s(c)dc\leq \tilde{z}\left[d^{-1}(\tilde{z})-s^{-1}(\tilde{z})\right]=\pi(\tilde{p},\tilde{w})$, where the latter inequality is due to that $\tilde{z}\leq\hat{z}$ and that $s^{-1}(z)$ is increasing in z. Moreover, by definition, $\pi(\tilde{p},\tilde{w})\leq\pi(p^*,w^*)$.

(e)-(f) Given the total matched supply and demand equal to z, the supply side surplus is equal to $\int_0^{s^{-1}(z)}s(c)dc$ and the demand side surplus is equal to $\int_{d^{-1}(z)}^{\bar{v}}d(v)dv$. Since $z^*\leq\tilde{z}\leq\hat{z}$ from part (c), we have the desired results by noting that $s^{-1}(z)$ is an increasing function and $d^{-1}(z)$ is a decreasing function.

(g) By definition, $W(\tilde{p},\tilde{w})\leq W(\hat{p},\hat{w})$. Moreover,

$$W(p^*,w^*)=U(p^*,w^*)+U^d(p^*,w^*)\leq U(\tilde{p},\tilde{w})+U^d(p^*,w^*)\leq U(\tilde{p},\tilde{w})+U^d(\tilde{p},\tilde{w})=W(\tilde{p},\tilde{w}),$$

where the first inequality holds because $\tilde{z}$ maximizes platform's profit plus supplier surplus, and the second inequality is due to part (e). □

Theorem 2.5 shows that a profit maximizing platform would set the highest price and the lowest wage. If the platform considers not only its own profit but also the surplus from independent suppliers, the platform would increase the wage paid out to the suppliers which in turn leads to more suppliers willing to provide the goods or services. To also utilize those additional suppliers, the platform has to lower the price charged to customers. If the platform further takes into account the consumer surplus to maximize the social welfare, the platform would further lower the price charged to customers and as more customers arrive to the market, the platform would further increase the wage paid out to suppliers. That is, with supplier surplus taken into consideration in setting price and wage, the platform would squeeze its own margin and as a result, the matching quantity increases. Interestingly, both supply and demand side benefit even though demand surplus is not in the platform's objective. With all parties' surplus taken into account, the platform would further squeeze its own margin and reaches the largest the matching quantity and the highest supply and demand surplus, whereas its own profit is the lowest.

Analogous results to maximizing the platform's profit plus supplier surplus can be obtained for the case where the platform's profit plus demand surplus is maximized.

Proposition 2.6 *Let the problem of maximizing the platform's profit plus demand surplus denote by* $\breve{}$. *All of the results in Theorem 2.5 hold with* $\tilde{}$ *replaced by* $\breve{}$.

Proof of Proposition 2.6. Let $\breve{z}$ be the optimal matching quantity that maximizes the platform's profit plus demand side surplus. We show that $z^*\leq\breve{z}\leq\hat{z}$. The other parts follow similarly to the proof of Theorem 2.5 and are hence omitted.

Consider $\tilde{U}(p,w)=U^d(p,w)+(p-w)[s(w)\wedge d(p)]=[s(w)\wedge d(p)]\left[p-w+\int_p^{\bar{v}}\bar{F}_Y(v)dv/\bar{F}_Y(p)\right]=[s(w)\wedge d(p)]\left[p-w+\int_p^{\bar{v}}d(v)dv/d(p)\right]$.

Define $\tilde{w} \equiv -w$ and $\tilde{p} \equiv -p$. Define $\tilde{s}(\tilde{w}) \equiv s(-\tilde{w})$ and $\tilde{d}(\tilde{p}) \equiv d(-\tilde{p})$.

Then, $\tilde{U}(p,w)$ can be rewritten as

$$\tilde{U}(p,w) = [\tilde{s}(\tilde{w}) \wedge \tilde{d}(\tilde{p})] \left[\tilde{w} - \tilde{p} + \int_{-\bar{v}}^{\tilde{p}} \tilde{d}(v)dv/\tilde{d}(\tilde{p})\right].$$

Let $\tilde{h}(\tilde{p}) = \tilde{p} - \int_{-\bar{v}}^{\tilde{p}} \tilde{d}(v)dv/\tilde{d}(\tilde{p})$. Then, $\tilde{h}'(\tilde{p}) = \tilde{d}'(\tilde{p}) \int_{-\bar{v}}^{\tilde{p}} \tilde{d}(v)dv/\tilde{d}(\tilde{p}) \geq 0$. Let $\tilde{w} = m + \tilde{h}(\tilde{p})$. To maximize $\tilde{U}(p,w) = m[\tilde{d}(\tilde{p}) \wedge \tilde{s}(m + \tilde{h}(\tilde{p}))]$, we must equate $\tilde{d}(\tilde{p})$ and $\tilde{s}(m + \tilde{h}(\tilde{p}))$. Let $\tilde{d}(\tilde{p}) = \tilde{s}(m + \tilde{h}(\tilde{p})) = z$. It follows that $\tilde{p} = \tilde{d}^{-1}(z)$ and $\tilde{w} = m + \tilde{h}(\tilde{d}^{-1}(z)) = \tilde{s}^{-1}(z)$. As a result, $m = \tilde{s}^{-1}(z) - \tilde{h}(\tilde{d}^{-1}(z)) = \tilde{s}^{-1}(z) - \tilde{d}^{-1}(z) + z^{-1} \int_{-\bar{v}}^{\tilde{d}^{-1}(z)} \tilde{d}(v)dv$.

To maximize $U(p,w)$, it is equivalent to solve

$$\max_{z\geq 0} \left\{ z\left[\tilde{s}^{-1}(z) - \tilde{d}^{-1}(z)\right] + \int_{-\bar{v}}^{\tilde{d}^{-1}(z)} \tilde{d}(v)dv \right\} = \max_{z\geq 0} \left\{ z\left[d^{-1}(z) - s^{-1}(z)\right] + \int_{d^{-1}(z)}^{\bar{v}} d(v)dv \right\}.$$

Let $\breve{z}$ be the minimum maximizer to the above problem.

To maximize its own profit, the platform solves $\max_{z\geq 0} z[d^{-1}(z) - s^{-1}(z)] = \max_{z\geq 0} z\left[\tilde{s}^{-1}(z) - \tilde{d}^{-1}(z)\right]$. It follows that $\tilde{z} \geq z^*$.

To maximize social welfare, one solves $\max_{z\geq 0}\left\{z\left[d^{-1}(z) - s^{-1}(z)\right] + \int_{d^{-1}(z)}^{\bar{v}} d(v)dv + \int_0^{s^{-1}(z)} s(c)dc\right\}$. Since $(d/dz)\int_{d^{-1}(z)}^{\bar{v}} d(v)dv \geq 0$ and $(d/dz)\int_0^{s^{-1}(z)} s(c)dc \geq 0$, we have $z^* \leq \breve{z} \leq \hat{z}$. □

2.6.2 Piecewise Commission Rate Contracts

Here we extend the flat, across-the-board commission contract to piecewise commission contracts.

Definition 2.3 ((Price-Based) Piecewise Contract) *A contract $w = f(p)$ is called a piecewise commission contract with $N+1$ pieces if there exist thresholds $t_1 < \cdots < t_N$ and constant rates $\gamma_1, \ldots, \gamma_{N+1} \in [0,1]$ such that $w = f(p) = \gamma_i p$ for $t_i \leq p < t_{i+1}$ $(i = 0, \ldots, N)$, with the convention $t_0 \equiv 0$ and $t_{N+1} \equiv \infty$.*

There always exists a piecewise commission contract that has K pieces (K is the number of scenarios) and is ϵ-optimal for any $\epsilon > 0$. This is because in the proof of Proposition 2.2, the constructed ϵ-optimal contract is in fact a piecewise contract. However, a K-piece commission contract would become impractical if the number of scenarios, K, is very large. Therefore, a reasonable number of pieces is desired to balance the complexity and performance of a contract. For instance, the fixed commission contract is effectively a 1-piece commission contract.

To study the piecewise commission contract, we consider the following commission structure that clusters the scenarios with a similar ratio $\gamma_k^* = w_k^*/p_k^*$.

Definition 2.4 (Ratio-Clustered Commission Structure) *A ratio-clustered commission structure with J pieces and thresholds $\bar{\gamma} = \theta_1 \geq \cdots \geq \theta_J \geq \underline{\gamma} \geq \theta_{J+1}$ is a commission structure that enforces $w = f(p) = \theta_j p$ for any scenario $k \in \mathcal{K}_j \equiv \{k \in \mathcal{K} \mid \theta_j \geq \gamma_k^* \geq \theta_{j+1}\}$.*

The following result investigates the number of clusters that is needed by a ratio-clustered commission structure to guarantee a near-optimality ratio δ.

Proposition 2.7 *For any $\delta \in (0,1)$, the ratio-clustered commission structure with $J = \lceil \log(\frac{1-\bar{\gamma}}{1-\underline{\gamma}})/\log\delta \rceil$ pieces and thresholds $\bar{\gamma} = \theta_1 \geq \cdots \geq \theta_J \geq \underline{\gamma} \geq \theta_{J+1}$ such that $\theta_{j+1} = 1-\delta^{-1}(1-\theta_j)$ (for all $j = 1, \ldots, J$) achieves the near-optimality ratio δ.*

Proof of Proposition 2.7. By definition, $\theta_j \geq \gamma_k^*$ for any $k \in \mathcal{K}_j$. It follows from the proof of Theorem 2.3 that $P_k^*(\theta_j)/\pi_k(p_k^*, w_k^*) \geq (1-\theta_j)/(1-\gamma_k^*) \geq (1-\theta_j)/(1-\theta_{j+1})$. Under the condition $\theta_{j+1} = 1-\delta^{-1}(1-\theta_j)$ (for all j), we have $1-\theta_{j+1} = \delta^{-1}(1-\theta_j)$, implying that $P_k^*(\theta_j)/\pi_k(p_k^*, w_k^*) \geq \delta$ for all $k \in \mathcal{K}$. Furthermore,

$$1-\theta_{J+1} = \delta^{-J}(1-\theta_1) = \delta^{-J}(1-\bar{\gamma}) = e^{-J\log\delta}(1-\bar{\gamma}) = e^{-\lceil \frac{\log[(1-\bar{\gamma})(1-\underline{\gamma})^{-1}]}{\log\delta} \rceil \log\delta}(1-\bar{\gamma}) \geq 1-\underline{\gamma},$$

which implies that $\theta_{J+1} \leq \underline{\gamma}$. Thus, the specified contract with $J = \lceil \log(\frac{1-\bar{\gamma}}{1-\underline{\gamma}})/\log\delta \rceil$ thresholds will achieve the pre-specified near-optimality ratio δ. □

We illustrate Proposition 2.7 by the following example.

Example 2.1 *Consider the case with $\bar{\gamma} = 0.8$ and $\underline{\gamma} = 0.2$. By Proposition 2.7, to achieve the near-optimality ratio $\delta = 0.5$, $J = \log(0.2/0.8)/\log(0.5) = 2$ pieces are needed. If $\bar{\gamma} = 0.9$ and $\underline{\gamma} = 0.1$, to achieve the same $\delta = 0.5$, $J = \lceil \log(0.1/0.9)/\log(0.5) \rceil = 4$ pieces are needed.* □

Given the optimal wage-to-price ratios $\gamma_{i_1} \geq \gamma_{i_2} \geq \cdots \geq \gamma_{i_N}$, if it happens that $p_{i_1}^* \leq \cdots \leq p_{i_N}^*$ or $p_{i_1}^* \geq \cdots \geq p_{i_N}^*$ (i.e., p_k^* is monotone in γ_k^*), the following proposition finds a piecewise fixed commission rate contract that achieves the near-optimality ratio δ.

Proposition 2.8 *Consider a ratio-clustered commission structure with J pieces and thresholds $\bar{\gamma} = \theta_1 \geq \cdots \geq \theta_J \geq \underline{\gamma} \geq \theta_{J+1}$ such that $\theta_{j+1} = 1-\delta^{-1}(1-\theta_j)$ for all j. If p_k^* is monotone in γ_k^*, there exist price thresholds $t_1 \geq \cdots \geq t_{J+1}$ such that the piecewise fixed commission rate contract $f(p) = \theta_j p$ for $t_{j+1} \leq p < t_j$ achieves the near-optimality ratio δ.*

Proof of Proposition 2.8. Suppose without loss of generality that $\gamma_1^* \geq \gamma_2^* \geq \cdots \geq \gamma_K^*$ and

$$\theta_1 \geq \gamma_1^* \geq \gamma_2^* \geq \cdots \geq \gamma_{i_1}^* \geq \theta_2,$$
$$\theta_2 \geq \gamma_{i_1+1}^* \geq \gamma_{i_1+2}^* \geq \cdots \geq \gamma_{i_2}^* \geq \theta_3,$$
$$\ldots,$$
$$\theta_J \geq \gamma_{i_{J-1}+1}^* \geq \cdots \geq \gamma_{i_J}^* \geq \theta_{J+1}.$$

We consider the case with $p_1^* \geq p_2^* \geq \cdots \geq p_K^*$ (the case with $p_1^* \leq p_2^* \leq \cdots \leq p_K^*$ can be dealt with similarly). We can find price thresholds $t_1 \geq \cdots \geq t_J$ such that

$$t_1 > p_1^* \geq p_2^* \geq \cdots \geq p_{i_1}^* \geq t_2,$$

$$t_2 > p^*_{i_1+1} \geq p^*_{i_1+2} \geq \cdots \geq p^*_{i_2} \geq t_3,$$
$$\ldots,$$
$$t_J > p^*_{i_{J-1}+1} \geq \cdots \geq p^*_{i_J} \geq t_{J+1}.$$

In the above, we implicitly assumed that $p^*_{i_1} > p^*_{i_1+1}$, $p^*_{i_2} > p^*_{i_2+1}$, …, and $p^*_{i_{J-1}} > p^*_{i_{J-1}+1}$. If this is not the case, say, $p^*_{i_1} = p^*_{i_1+1}$, we can replace $p^*_{i_1+1}$ with $p^*_{i_1+1} + \epsilon$; the resulting strategy will lead to an arbitrarily close performance as $\epsilon \to 0$.

Let us use the piecewise fixed commission rate contract $w = f(p) = \theta_j p$ for $p \in [t_{j+1}, t_j]$. In any Scenario k with $p^*_k \in [t_{j+1}, t_j)$ (p^*_k must belong to one of the intervals in $\{[t_{j+1}, t_j)\}_{j=1,\ldots,J}$), subject to the contract f, we have $\theta_j \geq \gamma^*_k \geq \theta_{j+1}$, and the optimal profit is

$$\begin{aligned}
P_k(f) =& \max_p [p - f(p)] \min\{s_k(f(p)), d_k(p)\} \\
\geq& \max_{t_{j+1} \leq p < t_j} [p - f(p)] \min\{s_k(f(p)), d_k(p)\} \\
=& \max_{t_{j+1} \leq p \leq t_j} (1-\theta_j) p \min\{s_k(\theta_j p), d_k(p)\} \\
\geq& (1-\theta_j) p^*_k \min\{s_k(\theta_j p^*_k), d_k(p^*_k)\} \\
\geq& (1-\theta_j) p^*_k \min\{s_k(\gamma^*_k p^*_k), d_k(p^*_k)\} \\
=& \frac{1-\theta_j}{1-\gamma^*_k}(1-\gamma^*_k) p^*_k \min\{s_k(\gamma^*_k p^*_k), d_k(p^*_k)\} \\
=& \frac{1-\theta_j}{1-\gamma^*_k} \pi_k(p^*_k, w^*_k) \\
\geq& \frac{1-\theta_j}{1-\theta_{j+1}} \pi_k(p^*_k, w^*_k) \\
=& \delta \pi_k(p^*_k, w^*_k),
\end{aligned}$$

where the last equality is due to the condition $\theta_{j+1} = 1 - \delta^{-1}(1-\theta_j)$. In any Scenario k, the piecewise contract f achieves at least a fraction δ of the optimal profit in the benchmark. Thus, overall, the piecewise contract achieves at least the fraction δ of the optimal expected profit in the benchmark. □

Below we present a special case in which p^*_k is indeed monotone in γ^*_k.

Example 2.2 (WHEN p^*_k IS MONOTONE IN γ^*_k) *Suppose that both supply and demand are linear functions in every Scenario k. Let $s_k(w) = \alpha_k w$ and $d_k(p) = d_{k0} - \beta_k p$. Then, in Scenario k, $p^*_k = d_{k0}\{[2(\alpha_k+\beta_k)]^{-1} + (2\beta_k)^{-1}\}$, $w^*_k = d_{k0}[2(\alpha_k+\beta_k)]^{-1}$ and $\gamma^*_k = w^*_k/p^*_k = \beta_k/(\alpha_k + 2\beta_k)$. γ^*_k is monotone in p^*_k, for example, when d_{k0} is increasing in k, both α_k and β_k are decreasing in k, and α_k/β_k is monotone in k. This may happen when, for example, in the ride-hailing problem, the weather condition worsens as the index k increases, so that the base demand d_{k0} becomes larger, and both the customers and suppliers become less sensitive to price (due to increased valuations and costs). Moreover, one of the sensitivity parameters α_k and β_k is more responsive to the change of k than the other.* □

If p_k^* is not monotone in γ_k^*, we can still construct a price-based piecewise commission contract that achieves the same near-optimality ratio δ. However, the former is likely to have more pieces than the latter. For example, consider $K = 6$, $\theta_1 \geq \gamma_1^* \geq \gamma_2^* \geq \gamma_3^* \geq \gamma_4^* \geq \theta_2 \geq \gamma_5^* \geq \gamma_6^* \geq \theta_3$ and $p_1^* \geq p_2^* \geq p_5^* \geq p_6^* \geq p_3^* \geq p_4^*$. We can find t_1, t_2, t_3, t_4 such that $t_1 \geq p_1^* \geq p_2^* \geq t_2 \geq p_5^* \geq p_6^* \geq t_3 \geq p_3^* \geq p_4^* \geq t_4$. We can then use the piecewise contract $f(p) = \theta_1 p$ for $t_2 \leq p < t_1$ and for $t_4 \leq p < t_3$, and $f(p) = \theta_2 p$ for $t_3 \leq p < t_2$. Following the same analysis as in the proof of Proposition 2.8, we can conclude that the piecewise contract achieves the near-optimality ratio δ if $\theta_{j+1} = 1 - \delta^{-1}(1 - \theta_j)$ for $j = 1, 2$.

2.6.3 Price-Setting Suppliers

We now consider the case in which prices are set by suppliers rather than by the platform. Suppose that the platform offers the commission contract $w = \gamma p$ to the suppliers with the commission rate $1 - \gamma$ determined. Then, each supplier simultaneously sets his/her own price, followed by all customers simultaneously make a purchase. As earlier, we consider continua of suppliers and customers, with the c.d.f.s of their opportunity cost and valuation denoted by F_X and F_Y, respectively, which are common knowledge. Individual supplier cost and customer valuation are private information. With possibly multiple prices, since the product is homogeneous, customers always prefer a lower price and their search is costless. If there is an imbalance between supply and demand who are willing to transact, some pre-announced rationing rule governs.

Again, we look at one individual scenario first, and so we omit the index k for the time being.

Proposition 2.9 (Decentralized Market) *Consider an arbitrary scenario. Given the commission γ, let $\bar{p}(\gamma)$ be the solution to $s(\gamma p) = d(p)$. There exists a symmetric Bayesian price equilibrium $p^*(c)$ for suppliers depending on their individual supply cost c:*

(i) $p^*(c) = \bar{p}(\gamma)$ for $c \leq \gamma\bar{p}(\gamma)$;

(ii) $p^*(c) = c$ for $c > \gamma\bar{p}(\gamma)$.

Moreover, in any equilibrium, all the suppliers that are able to sell will charge the price $\bar{p}(\gamma)$.

Proof of Proposition 2.9. Suppose that the suppliers price according to (i) and (ii). The condition $s(\gamma\bar{p}(\gamma)) = d(\bar{p}(\gamma))$ ensures that the suppliers with cost $c \in [0, \gamma\bar{p}(\gamma)]$ are completely matched with the demand with valuation $v \in [\bar{p}(\gamma), \infty)$. Any demand with $v < \bar{p}(\gamma)$ or supply with $c > \gamma\bar{p}(\gamma)$ are unmatched. We show that under the price schedule $p^*(c)$, no supplier has incentive to deviate.

Any supplier with $c \in [0, \gamma\bar{p}(\gamma)]$ is matched for sure, thus has no incentive to lower the price. If the supplier increase the price to $p' > \bar{p}(\gamma)$, he would be undercut by suppliers with $c \in (\bar{p}(\gamma), p')$ and thus left unmatched. Therefore, the supplier has no incentive to increase price either.

For a supplier with $c > \gamma\bar{p}(\gamma)$, his/her surplus is zero. Further increasing the price would still lead to zero surplus since the supplier remains unmatched, and lowering the price below $p^*(c) = c$ would lead to a nonpositive surplus.

Consequently, no supplier has an incentive to deviate from $p^*(c)$.

Next, we show that in any price equilibrium, all the matched suppliers will set the same price. To see this, first note that if a supplier with price p can sell, any supplier with price $p' < p$ can also sell. However, for any supplier charging p', by increasing the price to $p - \epsilon$, he will still sell and earn a higher surplus.

It remains to show that the price p for all suppliers that sell must be $\bar{p}(\gamma)$. If $p < \bar{p}(\gamma)$, there are more customers willing to buy than suppliers willing to sell. Then, any supplier can increasing his/her price by ϵ and still sells. This leads to a higher surplus. If $p > \bar{p}(\gamma)$, there are fewer customers willing to buy than suppliers willing to sell. Thus, the suppliers who charges p sells with a probability less than 1. However, if one of them decreases the price by ϵ, then his/her probability of selling becomes 1, which increases his/her surplus. As a result, the equilibrium price must be $\bar{p}(\gamma)$. □

Proposition 2.9 says that in a decentralized market, in equilibrium, the market price is a market clearing price such that the total mass of suppliers who are willing to sell is equal to the total mass of customers who are willing to purchase and no rationing is needed. In contrast, in the centralized market where the platform sets the price, by Theorem 2.1, for a given γ, the optimal centralized price is $p^*(\gamma) = \max\{\bar{p}(\gamma), p^o\}$, where $p^o \in \arg\max_{p\geq 0}(p-\gamma p)d(p) = \arg\max_{p\geq 0}(1-\gamma)pd(p) = \arg\max_{p\geq 0} pd(p)$. If the self-interested platform sets the optimal price as p^o, then there are more suppliers who are willing to sell than customers; in a decentralized market, because there is potentially more supply than demand at the price p^o, individual suppliers will drive down the market price and more suppliers and customers will be matched. In this case, we see that the decentralized market generates higher social welfare than the situation where a self-interested centralized platform sets the price.

The platform's problem of maximizing profit in a given scenario is as follows:

$$\begin{aligned} \max \quad & (1-\gamma)\bar{p}(\gamma)d(\bar{p}(\gamma)) \\ \text{s.t.} \quad & 0 \leq \gamma \leq 1. \end{aligned}$$

The following result shows that the equilibrium price and wage coincide with the optimal price and wage when the platform sets both of them to maximize profit.

Proposition 2.10 *The optimal fixed commission that maximizes the platform's profit is given by* $\gamma^* = s^{-1}(z^*)/d^{-1}(z^*)$ *and the equilibrium price corresponding to the commission* γ^* *is* $p^* = d^{-1}(z^*)$*, where* z^* *is the optimal solution to* $\max_z z[d^{-1}(z) - s^{-1}(z)]$.

Proof of Proposition 2.10. For any supplier with $c \in [0, \gamma\bar{p}(\gamma)]$, his surplus is $\gamma\bar{p}(\gamma) - c$.

For any customer with $v \geq \bar{p}(\gamma)$, her surplus is $v - \bar{p}(\gamma)$. The total surplus on the demand side is $d_0 \int_{\bar{p}(\gamma)}^{\infty}[y - \bar{p}(\gamma)]f_Y(y)dy = d_0 \int_{\bar{p}(\gamma)}^{\infty} \bar{F}_Y(y)dy = \int_{\bar{p}(\gamma)}^{\infty} d(y)dy$.

Let $z = d(\bar{p}(\gamma))$. The equality $d(\bar{p}(\gamma)) = s(\gamma\bar{p}(\gamma))$ implies that $z = s(\gamma d^{-1}(z))$, or equivalently, $\gamma = s^{-1}(z)/d^{-1}(z)$. Given that $s(w)$ is increasing and $d(p)$ is decreasing, γ is increasing in z. The constraint $0 \leq \gamma \leq 1$ is equivalent to $0 \leq z \leq \bar{z}$, where $\bar{z}$ is determined by the equation $d^{-1}(\bar{z}) = s^{-1}(\bar{z})$. Substituting $\bar{p}(\gamma)$ with $d^{-1}(z)$ and γ with $s^{-1}(z)/d^{-1}(z)$, the objective function for maximizing profit becomes $z[d^{-1}(z) - s^{-1}(z)]$. Thus, the maximizing profit is equivalent to $\max_{z \geq 0} z[d^{-1}(z) - s^{-1}(z)]$. Note that $z \leq \bar{z}$ will be naturally satisfied because the profit becomes negative otherwise. Let z^* be optimal solution. Then $\gamma^* = s^{-1}(z^*)/d^{-1}(z^*)$. □

With a given commission, the outcome from the decentralized market could in general be different from the situation where a self-interested centralized platform sets the price, leading to a revenue loss for the platform. Proposition 2.10 says that if there is only one scenario, however, the platform can perfectly recover the profit-maximizing outcome by pre-setting the commission.

When there are multiple scenarios, the self-interested platform can still tilt the market more towards profit-maximization by optimizing the commission. In particular, with price-setting suppliers, the platform faces the problem $\max_{\gamma \in [0,1]} (1-\gamma) \sum_{k \in \mathcal{K}} \rho_k \bar{p}_k(\gamma) d(\bar{p}_k(\gamma))$ to maximize its expected profit. In contrast, with price-taking suppliers, the platform solves

$$\max_{\gamma \in [0,1]} (1-\gamma) \sum_{k \in \mathcal{K}} \rho_k \max_{p \geq 0} p \min\{s_k(\gamma p), d_k(p)\}.$$

In general, for a given γ, $\bar{p}(\gamma)$ does not maximize $p \min\{s_k(\gamma p), d_k(p)\}$. Thus, the platform makes more profit if it sets the price instead of the suppliers for any given commission contract $w = \gamma p$. Our numerical analysis in the next section shows that the platform's profit achieved by the optimal commission contract and the corresponding supplier and customer surplus for the model with price-setting suppliers are on average very close to those for the model with price-taking suppliers.

2.7 Numerical Experiments

Maximizing Platform's Profit plus Supply Side Surplus. Consider the problem of maximizing the platform's profit plus suppliers' surplus. Let the total number of scenarios be fixed at $K = 48$. We generate 400 instances with normally distributed supplier cost and customer valuation as in Section 2.5.3, and compare the optimal expected value of the platform's profit plus suppliers' surplus (by choosing the optimal price and wage in every scenario) with the maximum expected value of the platform's profit plus suppliers' surplus among all fixed commission rate contracts. Table 2.6 shows the statistics on the fraction of optimality achieved.

Table 2.6: Statistics of the Performance of the Fixed Commission Contract for Maximizing Platform Profit plus Suppliers' Surplus: Normal Distributions

Maximum	Minimum	Mean	Median	Standard Deviation
98.10%	88.98%	93.32%	93.43%	1.30%

Next, we generate 400 random instances with lognormally distributed supplier cost and customer valuation. Following the notation in Subsections 2.5.1 and 2.5.3, the parameters are drawn as follows: $s_{k0} \sim \mathcal{U}[0,1]$, $\mu_{s,k} \sim \mathcal{U}[\log(10), \log(20)]$, $\sigma_{s,k} \sim \mathcal{U}[0.09975, 0.8326]$, $d_{k0} \sim \mathcal{U}[0,1]$, $\mu_{d,k} \sim \mathcal{U}[\log(10), \log(20)]$, $\sigma_{d,k} \sim \mathcal{U}[0.09975, 0.8326]$. Table 2.7 shows the statistics on the fraction of optimality achieved by the best fixed commission rate contract.

Table 2.7: Statistics of the Performance of the Fixed Commission Contract: Lognormal Distributions

Maximum	Minimum	Mean	Median	Standard Deviation
96.87%	82.46%	93.39%	93.60%	1.80%

Tables 2.6 and 2.7 shows that the fixed commission rate contract is able to achieve the most of the optimal surplus for the platform and the suppliers combined.

Maximizing Social Welfare. We now consider the problem of maximizing the total social welfare, and compare the optimal expected social welfare with the expected social welfare achieved by the best fixed commission rate contract.

We first generate 400 random instances with normally distributed supplier cost and customer valuation (in the same way as we do for the maximization of the platform profit plus suppliers' surplus), and present the statistics of the fraction of achieved optimality in Table 2.8.

Table 2.8: Statistics of the Performance of the Fixed Commission Contract: Normal Distributions

Maximum	Minimum	Mean	Median	Standard Deviation
94.33%	37.58%	72.05%	72.62%	9.93%

We then generate 400 random instances with lognormally distributed supplier cost and customer valuation (again in the same way as we do for the maximization of platform profit plus suppliers' surplus). The statistics are shown in Table 2.9.

Table 2.9: Statistics of the Performance of the Fixed Commission Contract: Lognormal Distributions

Maximum	Minimum	Mean	Median	Standard Deviation
94.14%	37.37%	72.47%	72.74%	9.45%

Tables 2.8 and 2.9 show that, for maximizing social welfare, the fixed commission rate contract can lead to less desirable performances (at the least, 37.58% for normal distributions and 37.37% for lognormal distributions). Nevertheless, it still achieves a decent portion of optimality (on average, 72.05% for normal distributions and 72.74% for lognormal distributions).

Price-Setting Suppliers. Consider $K = 10$ scenarios. Both supplier cost and customer valuation follow conditional normal distributions as in Subsection 2.5.1. The parameters are shown in Table 2.10. We compare the platform's profit under the profit-maximizing fixed commission contract for both the pricing-taking mode and price setting model. The optimal ratios γ_s^* and γ_t^* coincide, both equal to 0.605. The profit of the price-taking model under the optimal commission is 0.7117, and the optimal profit of the price-setting model is 0.7089. While the latter is slightly lower, it achieves 99.6% of the former.

The total combined surplus of suppliers and the platform for the price-setting model under the commission contract $w = \gamma_s^* p$ is 0.9446, slightly lower than the price-taking model with that equal to 0.9528 under the contract $w = \gamma_t^* p$.

The total social welfare for the price-setting model under the commission contract $w = \gamma_s^* p$ is 1.1567, slightly higher than the price-taking model with that equal to 1.1556 under the contract $w = \gamma_t^* p$.

Table 2.10: Parameters

Scenario	1	2	3	4	5	6	7	8	9	10
s_{k0}	0.9124	0.1886	0.7833	0.6852	0.8668	0.8783	0.3787	0.1458	0.8765	0.5258
$\mu_{s,k}$	10.11	27.05	18.55	29.37	12.62	15.97	27.77	14.97	19.62	29.04
$\sigma_{s,k}$	3.68	10.46	3.44	8.50	2.63	4.95	5.20	2.11	2.98	9.46
d_{k0}	0.0462	0.2002	0.8590	0.8311	0.8291	0.3742	0.5179	0.8450	0.5368	0.9242
$\mu_{d,k}$	14.53	25.95	15.51	11.58	11.68	27.80	13.04	28.75	17.90	28.22
$\sigma_{d,k}$	2.29	6.71	3.93	1.69	4.58	5.21	4.83	3.50	2.86	3.72
ρ_k	0.0442	0.1656	0.0906	0.0595	0.0836	0.0427	0.1314	0.1318	0.1493	0.1013

We further generate 400 instances with normally distributed supplier cost and customer valuation where the parameters are drawn in the same way as in Subsection 2.5.3. Surprisingly, the optimal platform profit, platform profit plus suppliers' surplus, and social welfare achieved by the optimal commission contract (that maximizes the platform's profit) under the price-setting model is very close to that under the price-taking model.

Let P_s^*, U_s and W_s denote the platform's profit, platform's profit plus supplier surplus and social welfare, respectively, under the optimal profit-maximizing fixed commission contract for the price-setting model. Likewise, let P_t^*, U_t and W_t denote the corresponding quantities for the price-taking model. We show the statistics on the ratios P_s^*/P_t^*, U_s/U_t and W_s/W_t in Table 2.11.

We further generate another 400 instances with lognormally distributed supplier cost and customer valuation (again, the parameters of the lognormal distributions are drawn in the same way as in Subsection 2.5.3), and report the statistics on the ratios P_s^*/P_t^*, U_s/U_t and W_s/W_t in Table 2.12. The optimal profits achieved by the price-setting model and the price-taking model are again very close. While the ratios U_s/U_t and W_s/W_t can be as low as 65.53% and 68.84%, respectively, on average, the price-setting model achieves 93.04% and 96.49% of U_t and W_t under the price-taking model.

Table 2.11: Statistics of the Ratios P_s^*/P_t^*, U_s/U_t and W_s/W_t: Normal Distributions

	Maximum	Minimum	Mean	Median	Standard Deviation
P_s^*/P_t^*	100%	96.88%	96.68%	99.83%	0.42%
U_s/U_t	100.02%	88.83%	98.48%	99.03%	1.74%
W_s/W_t	100.58%	95.46%	99.60%	100%	0.74%

Table 2.12: Statistics of the Ratios P_s^*/P_t^*, U_s/U_t and W_s/W_t: Lognormal Distributions

	Maximum	Minimum	Mean	Median	Standard Deviation
P_s^*/P_t^*	100%	90.06%	98.69%	99.08%	1.42%
U_s/U_t	100%	65.53%	93.04%	94.61%	6.27%
W_s/W_t	107.08%	68.84%	96.49%	97.90%	5.53%

2.8 Conclusion

Matching supply with demand is a core idea of operations management. In this chapter, we study the pricing problem in regulating demand and crowdsourced supply simultaneously for an on-demand matching platform. This problem can be viewed as performing revenue management on both sides of the market. We show that the joint price and wage optimization problem with the transaction volume as the minimum of supply and demand has a fundamental difference from the classic supply chain management settings and the two-sided market problem in the economics literature. This observation might have been lost if we had a more complicated formulation. With the most parsimonious model, our discovery justifies the unique perspective of operations management in studying the two-sided pricing and matching problem. Moreover, our work sheds light on the efficiency of the commonly adopted commission contract by showing that it achieves a guaranteed portion of the platform's optimal expected profit under full flexibility of optimally choosing both wage and price for every possible market condition. The result has similar flavor to the close-to-optimal performance of the long chain configuration studied in the process flexibility literature.

Our paper has the following limitations, which we hope to address in future research. First, for a given market condition, unmatched supply or demand is assumed to be lost. One may consider a more realistic, multiple-period dynamic setting in which the platform sets price and wage over time and unmatched supply and demand can be carried over from period to period. Second, for simplicity, we ignore the spatial dimension that is important for a ride-hailing platform. Setting a low price in the suburban area during the morning rush will increase the supply in the city after the rush. A future work can introduce the spatial dimension into the joint price and wage optimization under supply and demand uncertainty. Third, though our numerical experiments are comprehensive, they may differ from the reality. It would be interesting to identify empirically the demand and supply curves and then test our results numerically. Lastly, we ignore competition among platforms. In practice there can be more than one platform competing for both supply and demand. It can be fruitful to study the performance of the fixed commission contract under competition among on-demand matching platforms.

Chapter 3

Dynamic Type Matching: The General Framework

3.1 Introduction

Consider a firm that manages the process of matching supply with demand in a periodic-review fashion. There are multiple types of demand and supply, with a reward r_{ij} generated by matching one unit of type i demand and one unit of type j supply. At the beginning of each period, demand and supply of various types arrive in *random* quantities. The firm's problem is to decide how to match them and to what extent, so as to maximize the total discounted rewards minus costs, given that unmatched demand and supply will incur unit waiting and holding cost rates c and h, respectively, and will be carried over to the next period with carry-over rates $\alpha \in [0, 1]$ and $\beta \in [0, 1]$, respectively.

That is exactly the essence of the problem faced by many intermediaries who *centrally* manage matchings in the sharing economy. Operations management deals with serveral processes including that of matching supply with demand. There is a new form of such process that calls for active management—a sharing economy with *crowdsourced* supply. For example, carpooling platforms such as iCarpool and UberPool match a driver heading to a destination with several riders to the same destination (or in the same direction). Amazon crowdsources inventories of an identical item from third-party merchants to its warehouses, to fulfill online orders.[1] A nonprofit organization, United Network for Organ Sharing (UNOS), allocates donated organs to patients in need of transplantation. These popular business and nonprofit sharing-economy models are based on what academics often call a *two-sided market* (Rochet and Tirole 2006). In such a market, an information technology platform is developed and maintained by an intermediary firm to make sharing-economy activities possible. Three parties are involved, namely, an

[1] Amazon calls that an "inventory commingling program." A product ordered from Amazon or a third-party seller may not have originated from the original seller. The program gives Amazon the flexibility to ship products on the basis of their geographic proximity to customers, thus shortening delivery times and reducing shipping costs.

intermediary firm, the demand side and the supply side. In this structure, the intermediary organization matches demand and supply of *heterogeneous* types.

Other than dealing with heterogeneous types, the matching of demand and supply by an intermediary in the sharing economy can be extremely difficult for at least two more reasons. First, there are time-varying uncertainties on both the demand and supply sides, which may be out of the control of the intermediary. Second, arrived but unmatched demand and supply may leave the market over time. Economic theories use the tool of "price" to match demand with supply. While price does play an important role in many marketplaces, especially at the strategic and tactical levels, day-to-day or minute-to-minute operations often require more than price adjustment to achieve efficiency in practice. For example, in ridesharing, though Uber is well-known for its "surge pricing," the same rate applies to all rides at the same time regardless of their origin and destination; in other words, the rate at any given time is exogenous to geographic locations as "types" of riders and drivers. For another example, the allocation of donated organs in the United States does not involve prices at all. Given that prices are exogenous[2] or irrelevant, intervention at the operational level, by directly matching supply with demand of various types, provides an efficient way for the intermediary organization to allocate the crowdsourced supply across different types of demand. In summary, the intermediary has the task of matching exogenous streams of demand and supply types to maximize total profit or social welfare, taking into account that there will be time-dependent random arrivals of demand and supply in the future and that unmatched demand and supply need to be compensated and may abandon.

In this chapter, we formulate the intermediary firm's dynamic matching problem as a discrete-time stochastic dynamic program and analyze it for the structural properties of optimal matching policies and good heuristic policies. We obtain a set of *distribution-free* structural results provided that demand and supply distributions have finite means, and propose a heuristic for computation.

Using only matching rewards, we establish a *modified Monge condition* that specifies a dominance relation between two pairs of demand and supply types. The modified Monge conditions are sufficient and robustly necessary for the optimal matching policy to satisfy the following priority properties in the dynamic matching problem. First, for *any* two pairs of demand and supply types with one strictly dominating the other, it is optimal to prioritize the matching of the dominating pair over the dominated pair. Second, it is optimal to greedily match a *perfect pair* of demand and supply types that dominates all other pairs sharing its demand or supply type. The modified Monge condition generalizes the condition of a Monge sequence, discovered by Gaspard Monge in 1781, which guarantees a static and balanced transportation problem to be solved by a greedy algorithm (see Table 1 for comparisons). As a result of the priority properties, the optimal matching policy boils down to a match-down-to structure (instead of matching as much as possible in the greedy algorithm) when considering a specific pair of demand

[2]The pricing part of Uber's practice is at a higher level than the matching part. A higher price can encourage more drivers and discourage more riders to arrive at the market. Given the price is determined and announced, the matching decisions are made at the operational level after drivers and riders see the price and enter the market. Because our structural results are distribution free, they can be useful for the matching decisions in Uber's business practice as well.

and supply types, along the priority hierarchy. In fact, in the optimal policy, if *some* pair of demand and supply types is not matched as much as possible, *all* pairs that are strictly dominated by this pair should not be matched at all.

Table 3.1: Comparisons between Monge Sequence and Modified Monge Conditions

Monge Sequence	Modified Monge Condition
static, *deterministic* and *balanced* transportation problem	*dynamic*, *stochastic* and *unbalanced* matching problem
on a sequence	on pairs
sufficient and necessary	sufficient, and *robustly* necessary
a greedy algorithm: (1) priority property (2) *match as much as possible*	our result: (1) priority property (2) *match-down-to policy*

While two pairs of demand and supply types that share a common node may not be comparable under the modified Monge condition, the priority properties continue to hold for those pairs that indeed satisfy the modified Monge conditions, even when not all pairs are comparable. In addition, we provide bounds and heuristics for the general problem as follows.

As a heuristic method, we consider the deterministic counterpart of the stochastic dynamic problem for any period with t amount of remaining time in the horizon and any given levels of demand and supply; this can be written as a linear program with $O(n \times m \times T)$ variables. We show that the deterministic model provides an upper bound on the optimal total surplus of the stochastic model, and that it is asymptotically optimal to re-solve the linear program for the current period and state and apply the solution as a heuristic policy, when the the arrival rate of demand and supply of every type becomes increasingly large.

3.2 Literature Review

We illustrate the high-level positioning of our framework with Figure 3.1. The proposed dynamic-matching framework can be viewed as a generalization of two foundations of operations management, i.e., inventory management where the firm orders the supply centrally (Zipkin 2000), and revenue management where the firm regulates the demand side with a fixed supply side (Talluri and van Ryzin 2006), and of a combination of the two, i.e., joint pricing and inventory control (Chen and Simchi-Levi 2012). Unlike in inventory management and revenue management, the supply in the sharing economy is crowdsourced. It adds complexity beyond existing operations frameworks of stochastic inventory theory and revenue management.

More specifically, in connection with inventory management, our framework is closely related to the literature on inventory rationing, (see, e.g., Evans 1968, Veinott 1965, Ha 1997a,b, de Véricourt et al.

2002, Abouee-Mehrizi et al. 2012 and Abouee-Mehrizi et al. 2014), which considers a single supply type and multiple demand types, and allows demand from the less valuable types to be rejected (thus lost or delayed) in anticipation of future demand from the more valuable types. The matching decisions in our framework generalize the idea of inventory rationing by considering the characteristics of both the demand and supply types, such as marginal matching costs and abandonment rates. The current work is also related to a stream of research on production systems with random yield. Pioneered by Henig and Gerchak (1990), this stream considers unreliable production that yields only a random portion of the planned quantity. In contrast, our framework considers a class of problems with purely random sources of supply, independently of the firm's decisions, whereas the output from a random-yield production system is a random fraction or perturbation of the planned amount. In its connection with revenue management, our framework is closely related to quantity-based revenue management (see, e.g., Talluri and van Ryzin 2006, Part I), in particular, dynamic capacity allocation models with upgrading (see §4.1 of Chapter 4).

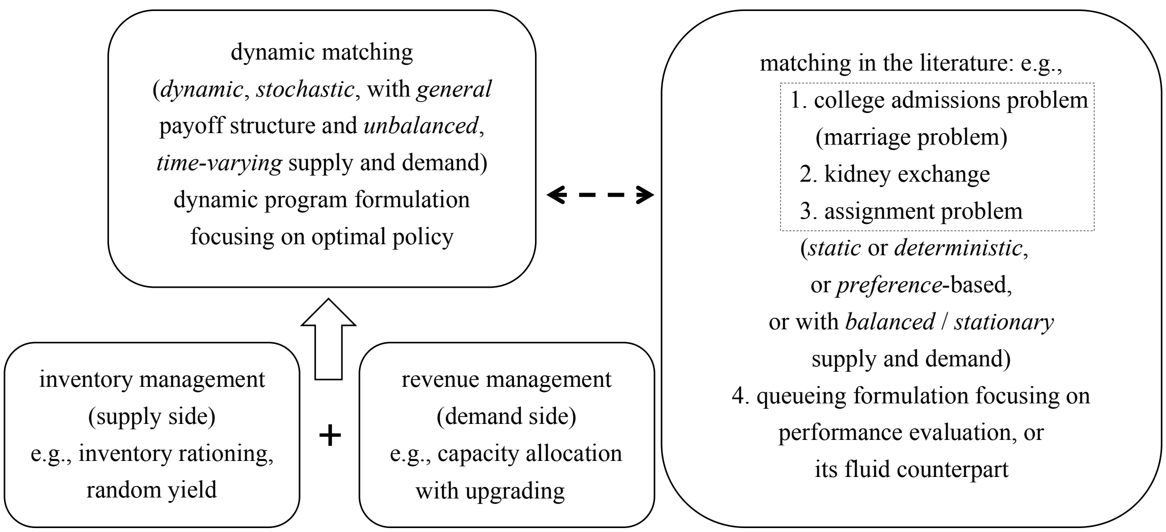


Figure 3.1: Positioning in the literature.

Driven by real-life applications, economists have studied the college admissions problem (with the marriage problem as a special case, see, e.g., Teo et al. 2001) and the kidney exchange problem. We compare our framework with those problems as follows.

The college admissions problem is *preference-based*, with the focus on the stable matchings. It involves parties on the demand and supply sides submitting preferences over options (see, e.g., Ashlagi and Shi 2014). As those matching outcomes such as marriage and college admissions can be life-changing, serious efforts in soliciting preferences are necessary. In contrast, as the sharing economy penetrates into our everyday lives, soliciting preferences may not be practical. For instance, when riders hail a car on Uber, they do not have the option, or may not even bother, to choose a driver to be matched with.

It requires the intermediary to associate pairs of demand and supply with rewards, as they arrive, and make matching decisions accordingly. To capture this situation, we assign a "monetary" contribution to a pair of demand and supply types. For example, a lower reward will be assigned if a farther-away car is dispatched. Moreover, the college admissions problem tends to have a *static* or *deterministic* nature. Supply and demand arrive with submitted preferences, before the matching decisions will be made, as in the classical marriage problem. In contrast, our framework, as in inventory and revenue management, emphasizes the *dynamic* and *stochastic* nature of a class of matching problems caused by the growth of the sharing economy and characterized by inter-temporal uncertainties.

Similarly to the college admissions problem, patients in kidney exchange have heterogeneous *preferences* over kidneys, subject to blood-type and tissue compatibility. (Note that kidney exchange is different from kidney allocation. In the latter, the organs are harvested from cadaveric donors; see below.) Moreover, in a typical situation the patient and donor arrive in pairs, with an incompatible (or less likely, compatible) patient and donor in each pair. Because of the compatibility issue and the fact that patients and donors arrive in pairs, efficient matching heuristics are focused on cycles, such as two-way exchanges or chains of patient-donor pairs; see, e.g., Roth et al. (2004, 2007). Most relevant to our framework is Ünver (2010), which studies dynamic kidney exchange with inter-temporal random arrivals of patient-donor pairs, and attempts to maximize the number of matched compatible pairs. In contrast, our model allows arbitrary *unbalanced* arrivals of demand and supply, with the objective to maximize social welfare or profit.

In addition to the above matching problems (i.e., college admissions and kidney exchange), serveral papers on dynamic matching mechanisms and/or decentralized dynamic matching are related to our work. For a finite-horizon decentralized matching model, Damiano et al. (2005) show that the equilibrium is characterized by an acceptance/participation threshold on the agent type. Baccara et al. (2016) consider the matching between two "rounds" and two "squares" that arrive over time (in each period, exactly one round and one square arrive), and compare the optimal centralized matching mechanism with the decentralized equilibrium. With stochastic arrival and departure of agents, Akbarpour et al. (2017) examine the performance of simple matching algorithms for minimizing loss of agents in large market limits. In contrast with those papers, we consider a centralized matching problem with general discrete-time arrival process, and maximize the total expected matching reward less costs.

Computer scientists has studied online bipartite matching problems, which have many applications such as allocation of display advertisements. Initiated by Karp et al. (1990), the classic version considers a bipartite graph $G = (U, V, E)$, and assumes that the vertices in U arrive in an "online" fashion. That is, only when a vertex $u \in U$ (e.g., a web viewer) arrives, are its incident edges (e.g., his interests) revealed. Then u can be matched to a previously unmatched adjacent vertex in V (e.g., an advertiser). The objective is to maximize the number of matchings. There are many variants, all with the focus on algorithms' competitive ratios (see Manshadi et al. 2012 for a more recent literature review). The main difference from our model is the "online" feature, other than that there is no clear notation of inventory, with one side (e.g., advertisers) always there and the other (e.g., impressions) getting lost if not matched. Instead of worst-case analysis, we focus on the expected value optimization.

Operations researchers have also been using the queueing approach or its fluid counterpart to study two-sided matching. With a fluid approach of modeling stochastic systems, Zenios et al. (2000) and Su and Zenios (2006) study kidney allocation by exploring the efficiency-equity trade-off, and Akan et al. (2012) study liver allocation by exploring the efficiency-urgency trade-off. Though focusing on structural properties of the optimal policy by exploring the stochastic dynamic program, we also propose a heuristic policy based on a fluid model, and show it is asymptotically optimal. Using double-sided queues, Zenios (1999) studies the transplant waiting list and Afèche et al. (2014) study trading systems like crossing networks. Su and Zenios (2004) analyze a queueing model with service discipline FCFS or LCFS to examine the role of patient choice in the kidney transplant waiting system. Adan and Weiss (2012) show that the stationary distribution of FCFS matching rates for two infinite multi-type sequences is of product form. These papers deal with performance evaluation under a given matching policy. Indeed, Gurvich and Ward (2014) study the dynamic control of matching queues with the objective of minimizing holding costs. The authors observe that in principle, the controller may choose to wait until some "inventory" of items builds up to facilitate more profitable matches in the future. We also make a similar observation.

3.3 The Model

We first introduce the notation before presenting our model. We use a boldface letter to denote a vector and its light face with subscript i to denote its i-th entry. By default, a vector is treated as a row vector. We also use a boldface letter to denote a matrix and its light face with subscript ij to denote its (i, j)-th entry. Let $\mathbf{x}_{[k,\ell]}$ denote the sub-vector of a vector $\mathbf{x}$, containing elements from the k-th entry to the ℓ-th entry, $\mathbf{e}^k_\ell$ the k-dimensional unit vector where the ℓ-th entry is 1 and all other entries are 0, and $\mathbf{e}^{n\times m}_{ij}$ the $n \times m$-dimensional matrix where the (i, j)-th entry is 1 and all other entries are 0. We denote by $\mathbf{1}^k$ a k-dimensional vector of 1's and denote by $\mathbf{0}^k$ a k-dimensional vector of 0's. (The superscript k may be omitted if the dimension of the zero vector is clear from the context). $\mathbb{R}_+ = \{r \mid r \geq 0\}$. $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. $x^+ = \max\{x, 0\}$ and $x^- = -\min\{x, 0\}$.

Consider a finite horizon with a total number of T periods. In practice, even though demand and supply arrive in continuous time, matching decisions are typically not made in real time. For example, Amazon periodically optimizes the way in which it matches customer orders and its warehouses (see Xu et al. 2009). At the beginning of each period, n types of demand and m types of supply arrive in *random* quantities. Let $\mathcal{D}$ be the set of demand types and $\mathcal{S}$ be the set of supply types. With a slight abuse of notation, we write $\mathcal{D} = \{1, 2, \ldots, n\}$ and $\mathcal{S} = \{1, 2, \ldots, m\}$, noting that $\mathcal{D}$ and $\mathcal{S}$ are disjoint sets.

We use i to index a demand type and j to index a supply type. The pairs of demand and supply are shown in Figure 3.2 as a bipartite graph. An arc (i, j) represents a match between type i demand and type j supply. For simplicity, we consider a complete bipartite graph in the base model. In other words, any demand type can potentially be matched with any supply type, obviously with different rewards (or equivalently, mismatch costs). We denote the complete set of arcs by $\mathcal{A} = \{(i, j) \mid i \in \mathcal{D}, j \in \mathcal{S}\}$.

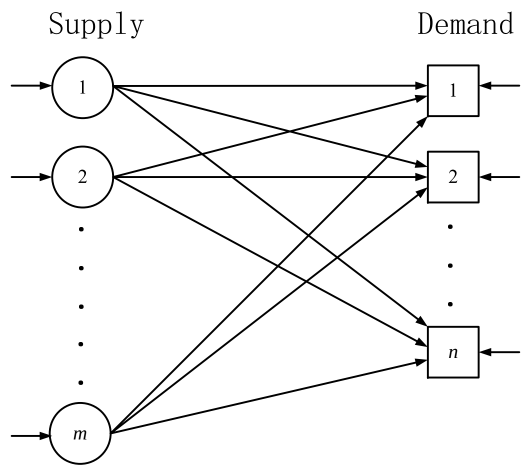


Figure 3.2: Pairs of demand and supply.

The state for a given period comprises the demand and supply levels of various types before matching but after the arrival of random demand $\mathbf{D}_t \in \mathbb{R}^n_+$ and supply $\mathbf{S}_t \in \mathbb{R}^m_+$ for that period. We make *no* strong assumption about the distributions of random demand and supply of various types. To make meaningful arguments on expectation, we do require that those distributions have bounded means, i.e., $ED_{it}, ES_{jt} < \infty$ for all i, j and any period t. In other words, our model and its results can be considered as being *distribution-free*. Moreover, the distributions of one period can be *exogenously* correlated with another. But our model does not account for endogenized correlations among distributions of demand and supply, e.g., a driver's current pickup of a customer may affect future supply at the place where the driver drops off the customer. That is, we assume away the possible dependence of future distributions of demand and supply on the current matching decisions.

We denote, as system states, the demand vector by $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$ and the supply vector by $\mathbf{y} = (y_1, \ldots, y_m) \in \mathbb{R}^m_+$, where x_i and y_j are the quantity of type i demand and type j supply available to be matched. Although we assume that the states and the demand and supply arrivals are continuous quantities (and therefore so are the matching decisions), our results can be readily replicated if those quantities are discrete. On observing the state $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{n+m}_+$, the firm decides on the quantity q_{ij} of type i demand to be matched with type j supply, for any $i \in \mathcal{D}$ and $j \in \mathcal{S}$. For conciseness, we write the decision variables of matching quantities in a matrix form as $\mathbf{Q} = (q_{ij}) \in \mathbb{R}^{n \times m}_+$, with $\mathbf{Q}_i$ its i-th row (as a row vector) and $\mathbf{Q}^j$ its j-th column (as a column vector). We assume that there is a reward r_{ij} for matching one unit of type i demand and one unit of type j supply for all i, j. Similarly, we can write the rewards in a matrix form as $\mathbf{R} = (r_{ij}) \in \mathbb{R}^{n\times m}$. Thus the total reward from matching is linear in the matching quantities. That is, $\mathbf{R} \circ \mathbf{Q} \equiv \sum_{i=1}^{n}\sum_{j=1}^{m} r_{ij}q_{ij}$, where "$\circ$" gives the sum of elements of the Hadamard product of two matrices. The *post-matching levels* of type i demand and type j supply are given by $u_i = x_i - \mathbf{1}^m\mathbf{Q}_i^{\mathsf{T}} = x_i - \sum_{j'=1}^{m} q_{ij'}$ and $v_j = y_j - \mathbf{1}^n\mathbf{Q}^j = y_j - \sum_{i'=1}^{n} q_{i'j}$, respectively. That is, $\mathbf{u} = \mathbf{x} - \mathbf{1}^m\mathbf{Q}^{\mathsf{T}}$ and $\mathbf{v} = \mathbf{y} - \mathbf{1}^n\mathbf{Q}$. The post-matching levels cannot be negative; i.e., $\mathbf{u} \geq \mathbf{0}$, $\mathbf{v} \geq \mathbf{0}$.[3]

After the matching is done in each period, each unit of unmatched demand and supply incurs a holding cost c and h respectively. The cost for demand could be loss of goodwill or waiting costs. Consequently, the total holding cost amounts to $c\mathbf{1}^n\mathbf{u}^{\mathsf{T}} + h\mathbf{1}^m\mathbf{v}^{\mathsf{T}} = c\sum_{i=1}^{n} u_i + h\sum_{j=1}^{m} v_j$. The unmatched demand and supply carry over to the next period with carry-over rates α and β, respectively. In other words, $(1-\alpha)$ fraction of demand and $(1-\beta)$ fraction of supply leave the system. Without loss of generality, we assume they leave the system with zero surplus.

The firm's goal is to determine a matching policy $\mathbf{Q}^* = (q_{ij}^*)$ that maximizes the expected total discounted surplus. (Our perspective is the maximizing of social welfare. Alternatively, the formulation can account for profit maximization if r_{ij} is interpreted as the revenue collected from a matching, and c and h are interpreted as the penalty paid to demand and supply for showing up but without a successful match in a period.) Let $V_t(\mathbf{x},\mathbf{y})$ be the optimal expected total discounted surplus given that it is in period t and the current state is $(\mathbf{x},\mathbf{y})$. We formulate the finite-horizon problem by using the following stochastic dynamic program:

$$\begin{aligned} V_t(\mathbf{x},\mathbf{y}) &= \max_{\mathbf{Q}\in\{\mathbf{Q}\geq\mathbf{0}|\mathbf{u}\geq\mathbf{0},\mathbf{v}\geq\mathbf{0}\}} H_t(\mathbf{Q},\mathbf{x},\mathbf{y}), \\ H_t(\mathbf{Q},\mathbf{x},\mathbf{y}) &= \mathbf{R}\circ\mathbf{Q} - c\mathbf{1}^n\mathbf{u}^{\mathsf{T}} - h\mathbf{1}^m\mathbf{v}^{\mathsf{T}} + \gamma E V_{t+1}(\alpha\mathbf{u}+\mathbf{D}_t, \beta\mathbf{v}+\mathbf{S}_t), \end{aligned} \tag{3.1}$$

where $\gamma \leq 1$ is the discount factor. The boundary conditions are $V_{T+1}(\mathbf{x},\mathbf{y}) = 0$ for all $(\mathbf{x},\mathbf{y})$, without loss of generality. In other words, at the end of the horizon, all unmatched demand and supply leave the system with zero surplus.

If $\alpha = \beta = 0$, decisions made in different periods are independent of each other. In that case, the problem reduces to a single period problem. In the remaining of this chapter, we consider the case $\beta > 0$, and assume without loss of generality that $\beta = 1$.

The existence of a solution to the dynamic program (3.1) is resolved by the following proposition.

Proposition 3.1 *The functions $H_t(\mathbf{Q},\mathbf{x},\mathbf{y})$ and $V_t(\mathbf{x},\mathbf{y})$ are continuous and concave. There exists an optimal matching policy $\mathbf{Q}_t^*(\mathbf{x},\mathbf{y})$.*

Proof of Proposition 3.1. We prove this result by induction on t. Clearly, $V_{T+1}(\mathbf{x},\mathbf{y}) \equiv 0$ is continous and concave in $(\mathbf{x},\mathbf{y})$. We suppose $V_{t+1}(\mathbf{x},\mathbf{y})$ is continous and concave in $(\mathbf{x},\mathbf{y})$, and show that so is $V_t(\mathbf{x},\mathbf{y})$. First, because $V_{t+1}(\mathbf{x},\mathbf{y})$ is continous in $(\mathbf{x},\mathbf{y})$, $H_t(\mathbf{Q},\mathbf{x},\mathbf{y})$ is continous in $(\mathbf{Q},\mathbf{x},\mathbf{y})$. Moreover,

[3]For simplicity, without formal definitions, we will take the liberty of using consistent notation for the post-matching levels, with its corresponding matching decision. For example, if a matching decision is denoted by $\bar{\mathbf{Q}}$, its corresponding post-matching levels will be denoted by $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$.

because the set mapping from $(\mathbf{x}, \mathbf{y})$ to the set $\mathcal{R}(\mathbf{Q}; \mathbf{x}, \mathbf{y}) = \{\mathbf{Q} \mid \mathbf{Q} \geq 0, \mathbf{u} = \mathbf{x} - \mathbf{1}^m \mathbf{Q}^\intercal \geq 0, \mathbf{v} = \mathbf{y} - \mathbf{1}^n \mathbf{Q} \geq 0\}$ is compact-valued and continuous, by the maximum theorem, $V_t(\mathbf{x}, \mathbf{y})$ is continous in $(\mathbf{x}, \mathbf{y})$. Second, since the composition of a concave function and an affine function is still concave (Simchi-Levi et al. 2014, Proposition 2.1.3(b)), $V_{t+1}(\alpha \mathbf{u} + \mathbf{D}, \beta \mathbf{v} + \mathbf{S})$ is concave in $(\mathbf{Q}, \mathbf{x}, \mathbf{y})$ for any given $(\mathbf{D}, \mathbf{S})$. Then, $E_{(\mathbf{D},\mathbf{S})}[V_{t+1}(\alpha \mathbf{u} + \mathbf{D}, \beta \mathbf{v} + \mathbf{S})]$ is concave in $(\mathbf{Q}, \mathbf{x}, \mathbf{y})$. Then it is immediately clear that $H_t(\mathbf{Q}, \mathbf{x}, \mathbf{y})$ is jointly concave in $(\mathbf{Q}, \mathbf{x}, \mathbf{y})$, because all other terms except the last term in (3.1) are linear in $(\mathbf{Q}, \mathbf{x}, \mathbf{y})$. Because the set $\mathcal{R}(\mathbf{Q}; \mathbf{x}, \mathbf{y})$ is a polyhedron defined by a system of linear inequalities, and a fortiori, a convex set, and the concavity is preserved under maximization over a convex set (Simchi-Levi et al. 2014, Proposition 2.1.15(b)), we have $V_t(\mathbf{x}, \mathbf{y}) = \max_{\mathbf{Q} \in \mathcal{R}(\mathbf{Q}; \mathbf{x}, \mathbf{y})} H_t(\mathbf{Q}, \mathbf{x}, \mathbf{y})$ is concave.

The existence of an optimal matching policy $\mathbf{Q}_t^*(\mathbf{x}, \mathbf{y})$ follows from the continuity of the function $H_t(\mathbf{Q}, \mathbf{x}, \mathbf{y})$ and the compactness of $\mathcal{R}(\mathbf{Q}; \mathbf{x}, \mathbf{y})$ for a given $(\mathbf{x}, \mathbf{y})$. □

In general we expect the *state-dependent* optimal policy to be extremely complex. Next we characterize some of its structural properties.

3.4 Priority Properties of the Optimal Policy

One may expect some intuitive properties of the optimal matching policy, e.g., matching a "perfect" pair in some sense, as much as possible. We provide sufficient conditions for such properties. Since we aim to address a general problem that has random dynamics, the conditions would sufficiently guarantee those properties even for a static problem. Therefore, the conditions we will provide are on the reward matrix and independent of any other system parameters. These conditions will guarantee that certain priority structural properties will hold for the dynamic problem at *any* time and with *any* realized demand and supply. For succinctness, we may only present the definitions and results on one side of the market, analogous definitions and results can be easily stated and obtained for the other side by symmetry.

3.4.1 Modified Monge Partial Order of Arcs

To facilitate discussion, we define a *relation* "$\succeq$" between arcs as follows and will show later it is a *partial order*. First, we consider neighboring arcs in the bipartite graph (Figure 3.2).

Definition 3.1 (Modified Monge condition for arcs with a common node) *$(i, j) \succeq (i, j')$, if*

$$\begin{aligned} &(i) \quad r_{ij} \geq r_{ij'} \quad \textit{and} \\ &(ii) \quad r_{ij} + r_{i'j'} \geq r_{ij'} + r_{i'j} \quad \textit{for all } i' \in \mathcal{D}. \end{aligned} \tag{D}$$

(When $i' = i$, condition (D) holds automatically. It is easy to see that $(i, j) \succeq (i, j')$ holds automatically for $j' = j$.)

Condition (D) is reminiscent of the *Monge sequence*. Hoffman (1963) provides a necessary and suffi-

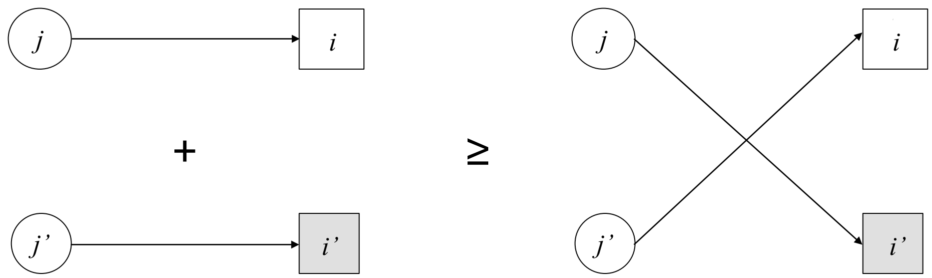


Figure 3.3: Condition (D).

cient condition for a transportation problem to be solvable by a greedy algorithm, in which a permutation (called a Monge sequence, and discovered by Gaspard Monge, a French mathematician, in 1781) is followed. A Monge sequence regulates *all* the arcs in the graph, requiring the inequality in condition (D) to hold *only* for all those neighboring arcs (i, j), (i, j') and (i', j) whenever (i, j) precedes (i, j') and (i', j) in the sequence. However, Definition 3.1 concerns *some* pairs of arcs but requires condition (D) to hold for *all* alternative nodes i' that are different from the common node i. The Monge sequence is introduced to solve a deterministic, demand-supply balanced transportation problem.We propose the partial order, termed as "modified Monge condition," to provide sufficient and robustly necessary conditions for structural priority properties in the dynamic demand-supply *unbalanced* matching problem with *random* inter-temporal demand and supply.

Part (i) of Definition 3.1 requires no less reward by matching pair (i, j) than pair (i, j'). To understand part (ii) of Definition 3.1, we compare the following two strategies: (1) matching one unit of type i demand and type j supply and another unit of type i' demand and type j' supply, and (2) matching one unit of type i demand and type j' supply and another unit of type i' demand and type j supply. The two strategies have the same post-matching levels of demand and supply. Condition (D) requires that the former strategy weakly dominate the latter (see Figure 3.3 for an illustration). In other words, part (ii) of Definition 3.1 implies that there does not exist $i' \in \mathcal{D}$ such that the latter strategy leads to a strictly higher reward than the former. As a result, part (ii) of Definition 3.1 eliminates the optimality of breaking up the pair (i, j) in matching nodes i, j and j'.

We further define a relation between arcs that do not share any node but can be connected through a sequence of neighboring arcs regulated by the relation "$\succeq$".

Definition 3.2 (Modified Monge condition for arcs without common nodes) *For $i \neq i'$ and $j \neq j'$, we say $(i, j) \succeq (i', j')$ if there exists a sequence of arcs $(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)$ such that either $i_k = i_{k+1}$ or $j_k = j_{k+1}$ for $k = 1, \ldots, n-1$, and $(i, j) = (i_1, j_1) \succeq (i_2, j_2) \succeq \cdots \succeq (i_k, j_k) = (i', j')$.*

In addition, the *equivalence* relation $(i, j) \simeq (i, j')$ means that $(i, j) \succeq (i, j')$ and $(i, j') \succeq (i, j)$ hold simultaneously. We say $(i, j) \succ (i', j')$ if $(i, j) \succeq (i', j')$ and $(i, j) \not\simeq (i', j')$. One can verify that the relation "$\succeq$" is a partial order over $\mathcal{A}$.

3.4.2 Priority Between Two Pairs of Supply and Demand Types

We will show if $(i,j) \succeq (i',j')$, (i,j) has a priority over (i',j') in the optimal dynamic matching. To show this, we first need the following lemma, which shows that a matching decision can be weakly improved by transferring quantity from the dominated arc (i',j') to the dominant one (i,j).

Lemma 3.1 *Suppose $(i,j) \succeq (i',j')$. In period t, if both decisions $\mathbf{Q}$ and $\mathbf{Q}+\epsilon\mathbf{e}_{ij}^{n\times m}-\epsilon\mathbf{e}_{i'j'}^{n\times m}$ are feasible for the state $(\mathbf{x},\mathbf{y})$, then $H_t(\mathbf{Q}+\epsilon\mathbf{e}_{ij}^{n\times m}-\epsilon\mathbf{e}_{i'j'}^{n\times m},\mathbf{x},\mathbf{y}) \geq H_t(\mathbf{Q},\mathbf{x},\mathbf{y})$. In other words, the decision $\mathbf{Q}+\epsilon\mathbf{e}_{ij}^{n\times m}-\epsilon\mathbf{e}_{i'j'}^{n\times m}$ weakly dominates $\mathbf{Q}$.*

Sketch of the proof. We sketch out the proof as follows. The full proof can be found in Section 3.8, along with the other proofs missing in the main body of this chapter. In period t, for a feasible decision $\mathbf{Q}$, if we transfer ϵ amount from (i',j') to (i,j) (i.e., decrease the matching quantity on (i',j') by ϵ and increase that on (i,j) by ϵ), the immediate benefit for the current period is $\epsilon(r_{ij}-r_{i'j'}) \geq 0$. However, as in any dynamic program, this transfer in the current period would also affect the initial states of the next period, hence also affecting all future periods. In particular, after the transfer, the post-matching levels (u_i,v_j) become $(u_i-\epsilon, v_j-\epsilon)$, and $(u_{i'},v_{j'})$ become $(u_{i'}+\epsilon, v_{j'}+\epsilon)$. To decide whether it is profitable to make the transfer now, one needs to evaluate its impact on all future periods. Suppose in a future period τ, there exists some type j'' supply that was supposed to be matched with i for an amount of $\tilde{\eta}_{j''}^{\tau}$ along a sample path. But now because type i demand could be in short due to the transfer, one may use i' instead. Such a replacement has an expected impact of $E(\tilde{\eta}_{j''}^{\tau})(r_{i'j''}-r_{ij''})$ for period τ. The following lemma suggests that the impact on the value functions due to the transfer from (i',j') to (i,j) in period $t-1$ is no worse than the sum of expected impacts due to replacements of i by i' and j by j' for all future periods from period t on. The proof is by induction.

Lemma 3.2 *In period t, for given $(\mathbf{x},\mathbf{y})$ with $x_i > 0$ and $y_j > 0$, $\epsilon_t^1 \in [0,x_i]$ and $\epsilon_t^2 \in [0,y_j]$, there exist $\eta_{j''}^{\tau} \geq 0$ and $\xi_{i''}^{\tau} \geq 0$ for $j'' \in \mathcal{S}$, $i'' \in \mathcal{D}$ and $\tau = t,\ldots,T+1$ such that $\sum_{\tau=t}^{T}\sum_{j''\in\mathcal{S}}\eta_{j''}^{\tau} \leq \epsilon_t^1$, $\sum_{\tau=t}^{T}\sum_{i''\in\mathcal{D}}\xi_{i''}^{\tau} \leq \epsilon_t^2$ and*

$$V_t(\mathbf{x}-\epsilon_t^1\mathbf{e}_i^n+\epsilon_t^1\mathbf{e}_{i'}^n,\mathbf{y}-\epsilon_t^2\mathbf{e}_j^m+\epsilon_t^2\mathbf{e}_{j'}^m)-V_t(\mathbf{x},\mathbf{y}) \geq \sum_{\tau=t}^{T}\gamma^{\tau-t}[\sum_{j''\in\mathcal{S}}\eta_{j''}^{\tau}(r_{i'j''}-r_{ij''})+\sum_{i''\in\mathcal{D}}\xi_{i''}^{\tau}(r_{i''j'}-r_{i''j})].$$

We further bound the sum of expected future impacts (i.e., the right hand side of the inequality in Lemma 3.2) from below by $-\epsilon(r_{ij}-r_{i'j'}) \leq 0$, which leads to the following result.

Lemma 3.3 *Suppose $(i,j) \succeq (i',j')$. In period t, for given $(\mathbf{x},\mathbf{y})$, $\epsilon_1 \in (0,x_i]$ and $\epsilon_2 \in (0,y_j]$, we have $V_t(\mathbf{x}-\epsilon_1\mathbf{e}_i^n+\epsilon_1\mathbf{e}_{i'}^n,\mathbf{y}-\epsilon_2\mathbf{e}_j^m+\epsilon_2\mathbf{e}_{j'}^m)-V_t(\mathbf{x},\mathbf{y}) \geq -\epsilon(r_{ij}-r_{i'j'})$, where $\epsilon = \max\{\epsilon_1,\epsilon_2\}$.*

Since the immediate benefit from the transfer is $\epsilon(r_{ij}-r_{i'j'})$ and by Lemma 3.3, the impact of the transfer on future periods can be bounded below by $-\epsilon(r_{ij}-r_{i'j'})$, the overall effect is nonnegative. □

Lemma 3.1 finds an improvement by transferring some matching quantity from the dominated arc (i',j') to the dominant arc (i,j). The result itself is reminiscent of the augmenting path approach to many network flow problems. For example, one can formulate a dynamic but *deterministic* transportation problem as a network flow problem (Bookbinder and Sethi 1980), which then can be solved by an augmenting path approach. However, in our dynamic matching problem with *random* future demand and supply, with a certain amount of "flow" transferred from (i,j) to (i',j') in period t, the state in the beginning of period $t+1$ will be changed. This requires matching quantities from period $t+1$ to the end of the horizon to change accordingly to remain feasible along a sample path. The change in period t (the transfer from (i',j') to (i,j)) and possible changes in periods $t+1,\ldots,T$ essentially form an "augmenting cycle," which contains directed arcs $i \to j$ and $j' \to i'$. Given the stochastic and dynamic nature of the problem, it is hard, if not impossible, to write the augmenting cycle in a simple, closed form for every sample path. Through backward induction, the proof of Lemma 3.1 quantifies the expected impact of possible changes in periods $t+1,\ldots,T$, and shows that the overall effect (together with the transfer of matching quantity in period t) is nonnegative. This approach adopts the idea of "augmenting path" for the stochastic dynamic program.

We need the following definitions to facilitate the presentation of our main result on the priority structure. For any arc $(i,j) \in \mathcal{A}$, we define a set of neighboring arcs that are *strictly* dominated by (i,j): $\mathcal{L}_{ij} \stackrel{\text{def}}{=} \{(i'',j) \mid (i,j) \succ (i'',j)\} \cup \{(i,j'') \mid (i,j) \succ (i,j'')\}$. We also define

$$w_{ij} = w_{ij}(\mathbf{Q},\mathbf{x},\mathbf{y}) \quad \stackrel{\text{def}}{=} \quad \min\{x_i - \sum_{j':(i,j')\notin\mathcal{L}_{ij}} q_{ij'},\ y_j - \sum_{i':(i',j)\notin\mathcal{L}_{ij}} q_{i'j}\}.$$

If $w_{ij}=0$, type i or j is exhausted by the matching over arcs outside the set $\mathcal{L}_{ij}$.

Theorem 3.1 (Partial order implies priority) *Without loss of generality, assume* $\mathbf{x} > \mathbf{0}$ *and* $\mathbf{y} > \mathbf{0}$ *in period* t.[4] *There exists an optimal decision* $\mathbf{Q}^*$ *such that for any* $(i,j) \succ (i',j')$, $\min\left\{w^*_{ij}, q^*_{i'j'}\right\} = 0$, *i.e., either* $\mathbf{Q}^*$ *exhausts type* i *or* j *over arcs outside* $\mathcal{L}_{ij}$, *or* $q^*_{i'j'} = 0$.

Proof of Theorem 3.1. Define $\mathcal{A}_1 = \{(i',j') \in \mathcal{A} \mid \nexists (i,j) \in \mathcal{A}$ such that $(i,j) \succ (i',j')\}$ as the set of undominated arcs. Further define $\mathcal{A}_k = \{(i',j') \in \mathcal{A}\backslash(\bigcup_{l=1}^{k-1}\mathcal{A}_l) \mid \nexists (i,j) \in \mathcal{A}\backslash(\bigcup_{l=1}^{k-1}\mathcal{A}_l)$ such that $(i,j) \succ (i',j')\}$ inductively. Since the total number of arcs is finite, only a finite number of $\mathcal{A}_k$'s are non-empty. Let $K+1 = \min\{k \in \mathbb{N} \mid \mathcal{A}_k = \emptyset\}$. Then $\mathcal{A} = \bigcup_{k=1}^{K}\mathcal{A}_k$.

For the given state $(\mathbf{x},\mathbf{y})$ and a feasible decision $\mathbf{Q}$, we construct another feasible decision that satisfies the desired priority property and weakly dominates $\mathbf{Q}$. Consider the following construction.

Step 0. Let $k \leftarrow 1$ and $\widetilde{\mathcal{A}} \leftarrow \mathcal{A}$.

Step 1. Pick $(i,j) \in \widetilde{\mathcal{A}} \bigcap \mathcal{A}_k$.

Step 2. Find a pair of arcs $(i'',j) \prec (i,j)$ and $(i,j'') \prec (i,j)$ such that $q_{i''j} > 0$ and $q_{ij''} > 0$. Let $\mathbf{Q} \leftarrow \mathbf{Q} - \epsilon\mathbf{e}^{n\times m}_{i''j} - \epsilon\mathbf{e}^{n\times m}_{ij''} + \epsilon\mathbf{e}^{n\times m}_{ij} + \epsilon\mathbf{e}^{n\times m}_{i''j''}$, where $\epsilon = \min\{q_{i''j}, q_{ij''}\}$. Repeat this step until we can no longer find such (i'',j) and (i,j''), at which point either $q_{i''j} = 0$ for all $(i'',j) \prec (i,j)$ or $q_{ij''} = 0$ for all $(i,j'') \prec (i,j)$.

Step 3. In the case in which $q_{i''j} = 0$ for all $(i'',j) \prec (i,j)$, find $(i,j'') \prec (i,j)$ such that $q_{ij''} > 0$ and

[4] If $x_i = 0$ (or $y_j = 0$), we can delete demand node i (or supply node j) and all its connected arcs, on which matching quantities are set to zero.

let $\mathbf{Q} \leftarrow \mathbf{Q} - \delta \mathbf{e}_{ij''}^{n\times m} + \delta \mathbf{e}_{ij}^{n\times m}$, where $\delta = \min\{q_{ij''}, v_j\}$. Repeat this until either $v_j = 0$ or $q_{ij''} = 0$ for all $(i'', j) \preceq (i, j)$.

In the case in which $q_{ij''} = 0$ for all $(i, j'') \prec (i, j)$, find $(i'', j) \prec (i, j)$ such that $q_{i''j} > 0$. Let $\mathbf{Q} \leftarrow \mathbf{Q} - \theta \mathbf{e}_{i''j}^{n\times m} + \theta \mathbf{e}_{ij}^{n\times m}$, where $\theta = \min\{q_{i''j}, u_i\}$. Repeat this until either $u_i = 0$ or $q_{i''j} = 0$ for all $(i, j'') \prec (i, j)$.

At the end of Step 3, one of the followings is true: (i) $u_i v_j = 0$; (ii) $q_{i''j} = q_{ij''} = 0$ for all $(i'', j) \prec (i, j)$ and $(i, j'') \prec (i, j)$.

Step 4. Find $(i', j') \prec (i, j)$ such that $q_{i'j'} > 0$. Let $\mathbf{Q} \leftarrow \mathbf{Q} - \eta \mathbf{e}_{i'j'}^{n\times m} + \eta \mathbf{e}_{ij}^{n\times m}$, where $\eta = \min\{u_i, v_j, q_{i'j'}\}$. Repeat this until either $u_i v_j = 0$ or $q_{i'j'} = 0$ for all $(i', j') \prec (i, j)$.

Step 5. Let $\widetilde{A} \leftarrow \widetilde{\mathcal{A}} \setminus \{(i, j)\}$. If $\widetilde{A} \bigcap \mathcal{A}_k = \emptyset$, let $k \leftarrow k + 1$. Go to Step 1 if $k \leq K$. Stop if $k > K$.

It is easy to see that each $(i, j) \in \mathcal{A}$ is chosen exactly once in Step 1. At the end of Step 2, suppose without loss of generality, that $q_{i''j} = 0$ for all $(i'', j) \prec (i, j)$. Then at the end of Step 3, either $v_j = 0$ or $q_{ij''} = q_{i''j} = 0$ for all $(i, j''), (i'', j) \in \mathcal{L}_{ij}$. In the former case, $y_j - \sum_{(i'',j)\notin\mathcal{L}_{ij}} q_{i''j} = y_j - \sum_{i''\in\mathcal{D}} q_{i''j} = v_j = 0$, implying that $w_{ij} = 0$. In the latter case, at the end of Step 4, either $q_{i'j'} = 0$ for all $(i', j') \prec (i, j)$ which satisfies the desired property, or $u_i v_j = 0$. For the case of $u_i v_j = 0$, $w_{ij} = \min\left\{x_i - \sum_{(i,j'')\notin\mathcal{L}_{ij}} q_{ij''}, y_j - \sum_{(i'',j)\notin\mathcal{L}_{ij}} q_{i''j}\right\} = \min\left\{x_i - \sum_{j''\in\mathcal{S}} q_{ij''}, y_j - \sum_{i''\in\mathcal{D}} q_{i''j}\right\} = \min\{u_i, v_j\} = 0$, where the second equality is due to $q_{ij''} = q_{i''j} = 0$ for all $(i, j''), (i'', j) \in \mathcal{L}_{ij}$. Thus, the desired property will be satisfied by any $(i', j') \prec (i, j)$ for a given (i, j). At the end of the whole construction procedure, the desired property will be satisfied by any pair $(i, j) \succ (i', j')$.

By Lemma 3.1, the construction procedure keeps weakly improving the matching decision via Steps 2, 3 and 4. Moreover, the procedure stops in a finite number of steps. In the end, we obtain a new feasible decision that satisfies the desired property and weakly dominates the original decision. □

The northwest corner rule under the assumption of a Monge sequence can completely solve the deterministic and balanced version of those problems in a greedy fashion. For the stochastic version, we show in Theorem 3.1 that the priority structure preserves under the modified Monge conditions, a somewhat stronger set of assumptions than the Monge sequence.[5] However, even a pair has higher priority in the optimal matching, they are not necessarily matched in a greedy fashion; when they are not exhausted, all pairs that have strictly lower priority should not be matched.

The condition $(i, j) \succ (i', j')$ is not necessary for the priority property; see Example 3.1 below. Nevertheless, it is "necessary" in a *robust* sense against all possible scenarios. In other words, if $(i, j) \succ (i', j')$ fails to hold, one can construct a scenario, by choosing the parameters other than the reward matrix $\mathbf{R}$, such that (i', j') has a higher priority over (i, j) in the optimal policy. Hence, the modified Monge conditions are arguably the best conditions on the reward matrix one can hope for in order to guarantee a general priority structure in the optimal policy.

Example 3.1 *Consider $\mathcal{D} = \{1, 2, 3\}$ and $\mathcal{S} = \{1, 2, 3\}$. Let $r_{13} = r_{22} = r_{31} = r_{33} = \epsilon$, $r_{12} = r_{21} = N$,*

[5] If all arcs are comparable under our partial order along the sequence, then it is a Monge sequence. But we do not require all arcs to be comparable in general.

$r_{11} = \frac{3}{2}N$, $r_{23} = r_{32} = 2N$, $c = h = \epsilon$, where ϵ is sufficiently small and N is sufficiently large. In the current period, assume $\mathbf{x} = \mathbf{y} = (1, 1, 0)$. See Figure 3.4 for an illustration. Suppose that there is a high chance of type 3 demand or supply arriving in the next period. It is optimal to save the unit of type 2 demand and the unit of type 2 supply for the future; i.e., $q^*_{2j'} = q^*_{i'2} = 0$ for all $i' \in \mathcal{D}$, $j' \in \mathcal{S}$. On the other hand, it is optimal to fully match the unit of type 1 demand and the unit of type 1 supply, i.e., $q^*_{11} = 1$. Thus, it is optimal to prioritize matching type 1 demand and type 1 supply over matching type 1 demand and type 2 supply. However, here $r_{11} + r_{22} < r_{12} + r_{21}$, implying that $(1, 1) \succ (1, 2)$ is false. ☐

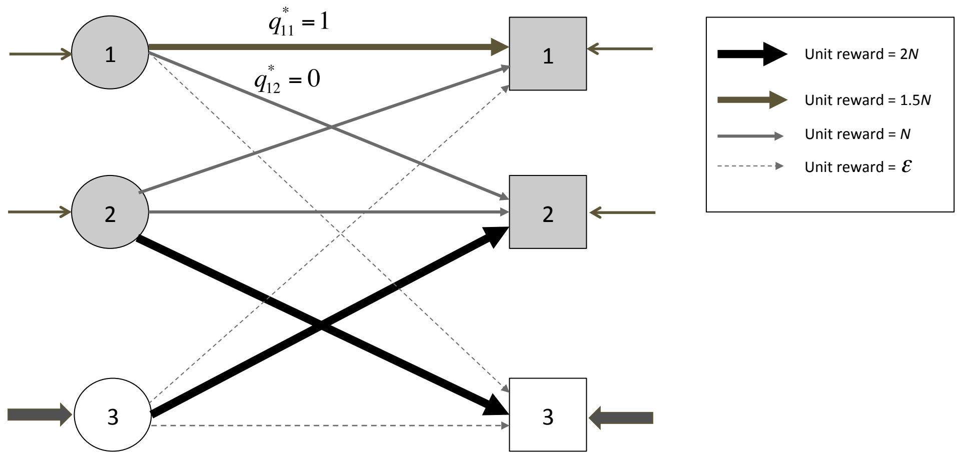


Figure 3.4: The structure of the matching problem in Example 3.1

3.4.3 Perfect Pair

Next we provide a sufficient condition for a pair of demand and supply types to be matched in a greedy fashion in preference to all other possible matching options.

Theorem 3.2 (When greedy matching is optimal) *If $(i, j) \succeq (i, j')$ for all $j' \in \mathcal{S}$ and $(i, j) \succeq (i', j)$ for all $i' \in \mathcal{D}$, then $q^*_{ij} = \min\{x_i, y_j\}$.*

Theorem 3.2 is *not* a direct consequence of Theorem 3.1. By directly applying Theorem 3.1, we can only say that under the conditions in Theorem 3.2, it is optimal for the firm to prioritize the matching of type i demand and type j supply over any other possibilities. However, it may still be possible that the firm has reserved some type i demand and type j supply without greedily matching them.

The conditions in Theorem 3.2, i.e., $(i, j) \succeq (i, j')$ for all j' and $(i, j) \succeq (i', j)$ for all i', say that the pair (i, j) dominates all other pairs that share type i demand or type j supply. We say that such a pair forms a *perfect pair* in the eyes of the intermediary firm. Example 3.1 also serves as a counterexample illustrating that conditions in Theorem 3.2 are not necessary for greedy matching, though one can say that they are "necessary" in a *robust* sense against all possible scenarios. The dominance relations in Theorem 3.2 contain two sets of conditions on rewards. The first set, $r_{ij} \geq \max_{i' \in \mathcal{D}, j' \in \mathcal{S}} \{r_{ij'}, r_{i'j}\}$, says that the matching between type i demand and type j supply generates the highest reward among other possible uses of those resources. As a result, type i demand and type j supply are the most favorable for each other from their own perspective. However, they may not form a perfect pair from the intermediary's point of view unless another set of conditions is satisfied. The following example illustrates that the condition $r_{ij} \geq \max_{i' \in \mathcal{D}, j' \in \mathcal{S}} \{r_{ij'}, r_{i'j}\}$ is not enough for the intermediary firm to adopt a greedy match. This is because from a centralized planner's perspective, the components of a most favorable pair for each other may be separately paired with others to generate an overall higher reward. This example emphasizes the importance of the second set of conditions in the modified Monge partial order—i.e., condition (D) holds for all i' with any given j' and for all j' with any given i'—for guaranteeing that a greedy match between type i demand and type j supply will be optimal.

Example 3.2 *The claim in Theorem 3.2 may fail without the set of condition (D)'s even for a single-period model. To see this, consider a one-period example with* $\mathcal{D} = \{1, 2, 3\}$ *and* $\mathcal{S} = \{1, 2, 3\}$*. Suppose that* $r_{11} = r_{22} = r_{33} = 2N$*,* $r_{12} = r_{21} = r_{23} = r_{32} = N + \epsilon$*,* $r_{13} = r_{31} = \epsilon$*, where* $N > \epsilon > 0$*. Here,* $r_{22} \geq \max\{r_{21}, r_{23}, r_{12}, r_{32}\}$*, i.e.,* $(2, 2)$ *generates the highest reward. In the current period, assume* $\mathbf{x} = (1, 1, 0)$ *and* $\mathbf{y} = (0, 1, 1)$*. If we fully match the type 2 demand with the type 2 supply, then the type 1 demand has to be matched with the type 3 supply given there is only one period, leading to a total reward of* $r_{22} + r_{13} = 2N + \epsilon$*. Alternatively, if we match the type 2 demand with the type 3 supply and match the type 1 demand with the type 2 supply, the total reward is* $r_{23} + r_{12} = 2(N + \epsilon)$*, which is higher than* $r_{22} + r_{13}$*, violating the condition* $(2, 2) \succeq (2, 3)$*. In this example, we see that although the type 2 demand and type 2 supply are the most favorable for each other in terms of generating the highest reward, they are not a perfect pair in the eyes of the centralized planner.* □

As an immediate application of Theorem 3.2, consider demand and supply types that are specified by their locations in an Euclidean space. The reward of matching supply with demand is a fixed prize minus the disutility proportional to the Euclidean distance between the supply location and the demand location. It is easy to verify that a demand type and a supply type from the *same* location forms a perfect pair, and by Theorem 3.2, they should be matched as much as possible. To see why they are a perfect pair, we have $r_{ii} + r_{i'j'} \geq r_{ij'} + r_{i'i}$ because $d_{i'j'} \leq d_{ij'} + d_{i'i}$, where d_{ij} is the Euclidean distance between the locations of type i demand and type j supply. The latter inequality is simply the triangle inequality. We summarize this result as follows.

Corollary 3.1 *In an Euclidean space with horizontally differentiated types as locations, it is optimal to greedily match the demand and supply from the same location.*

Corollary 3.1 suggests that with geographic locations as types, the intermediary firm such as Uber and Amazon should always match a demand with a supply if they are originated from the same geographic region, or practically speaking, if they are sufficiently close to each other.

The matching quantity between an imperfect pair needs to be find out numerically. In the next section, we propose a heuristic for tractable computations.

3.5 Bound and Heuristic

In this section we study the deterministic counterpart of the stochastic problem in its general form. We show that the heuristic suggested by the deterministic model can be computed efficiently and is asymptotically optimal for the stochastic problem.

3.5.1 The Deterministic Resolving Heuristic

We consider the deterministic model by ignoring the uncertainty and assume that the mean demand quantity $\lambda_{it} = ED_{it}$ and mean supply quantity $\mu_{jt} = ES_{jt}$ arrive in each period. The linear program

$$
\begin{aligned}
(\mathrm{P}_\tau^{\mathbf{x},\mathbf{y}}) \quad \max_{q_{ijt},x_{it},y_{jt}} \quad & \sum_{t=\tau}^{T}\gamma^{t-1}[\sum_{i=1}^{n}\sum_{j=1}^{m} r_{ij}q_{ijt} - c(\sum_{i=1}^{n} x_{it} - \sum_{i=1}^{n}\sum_{j=1}^{m} q_{ijt}) - h(\sum_{j=1}^{m} y_{jt} - \sum_{i=1}^{n}\sum_{j=1}^{m} q_{ijt})] \\
\text{s.t.} \quad & \sum_{j=1}^{m} q_{ijt} \le x_{it}, \quad i \in \mathcal{D},\ t = \tau, \tau+1, \ldots, T, \\
& \sum_{i=1}^{n} q_{ijt} \le y_{jt}, \quad 1 \le j \le m,\ \tau \le t \le T, \\
& x_{i,t+1} = \alpha(x_{it} - \sum_{j=1}^{m} q_{ijt}) + \lambda_{it}, \quad i \in \mathcal{D},\ t = \tau, \ldots, T-1, \\
& y_{j,t+1} = \beta(y_{jt} - \sum_{i=1}^{n} q_{ijt}) + \mu_{jt}, \quad 1 \le j \le m,\ \tau \le t \le T-1, \\
& q_{ijt} \ge 0, \quad i \in \mathcal{D},\ j \in \mathcal{S},\ t = \tau, \ldots, T, \\
& x_{i\tau} = x_i, \quad y_{j\tau} = y_j, \quad i \in \mathcal{D},\ j \in \mathcal{S}.
\end{aligned} \tag{3.2}
$$

gives the formulation of the problem from period τ to period T, where $(\mathbf{x}, \mathbf{y}) = (x_1, \ldots, x_n, y_1, \ldots, y_m)$ is a given initial state at the beginning of period τ.

From the optimal solution to $(\mathrm{P}_\tau^{\mathbf{x},\mathbf{y}})$, $\{\hat{q}_{ijt}, \hat{x}_{it}, \hat{y}_{jt}\}_{i\in\mathcal{D},j\in\mathcal{S},t=\tau,\ldots,T}$, we obtain a feasible matching decision $\{\hat{q}_{ij\tau}\}_{i\in\mathcal{D},j\in\mathcal{S}}$ in period τ for state $(\mathbf{x}, \mathbf{y})$, and use it as a heuristic decision. If we start in period 1 with an initial state $(\mathbf{x}_1, \mathbf{y}_1)$, we will solve $(\mathrm{P}_1^{\mathbf{x}_1,\mathbf{y}_1})$ to obtain matching decisions $\{\hat{q}_{ij1}\}_{i\in\mathcal{D},j\in\mathcal{S}}$ in period 1. Given a realization of demand and supply in period 2 as $\mathbf{D}_2 = \mathbf{d}_2$ and $\mathbf{S} = \mathbf{s}_2$, respectively. The state in period 2 is then $(\mathbf{x}_2, \mathbf{y}_2) = (\alpha\hat{\mathbf{u}} + \mathbf{d}_2, \beta\hat{\mathbf{v}} + \mathbf{s}_2)$. We then solve $(\mathrm{P}_2^{\mathbf{x}_2,\mathbf{y}_2})$ to obtain the heuristic decision in period 2. We proceed until period T to obtain the heuristic decisions along a sample path of demand and supply realization. We refer to this heuristic as the deterministic resolving heuristic.

Intuitively, one would expect that the uncertainty in demand and supply in the stochastic model would result in lower expected surplus. One can think of the variables x_{it}, y_{jt} and q_{ijt} as the expected available quantity of type i demand, that of type j supply and the expected matching quantity of arc (i, j) between i and j in period t. While the stochastic problem requires constraints in (3.1) to be satisfied for each sample path, the deterministic model only requires the expected variables x_{it}, y_{jt} and

q_{ijt} to satisfy the constraints. Thus the deterministic model is a relaxation of the stochastic one and provides an upper bound on the stochastic model's optimal surplus.

Proposition 3.2 (Deterministic upper bound) *The deterministic model provides an upper bound on the optimal total surplus of the stochastic model.*

Proof of Proposition 3.2. Let Ω be the set of all sample paths of demand and supply realizations over the finite horizon, $\omega \in \Omega$ be a sample path, and $p(\omega)$ be the density at ω. We rewrite the stochastic model in the following form of a stochastic program.

$$
\begin{aligned}
(\mathrm{S}_\tau^{\mathbf{x},\mathbf{y}}) \quad \max \quad & \int_\Omega p(\omega) \sum_{t=\tau}^{T} \gamma^{t-1} \{ \sum_{i=1}^{n}\sum_{j=1}^{m} r_{ij} q_{ijt}(\omega) - c \sum_{i=1}^{n} [x_{it}(\omega) - \sum_{j=1}^{m} q_{ijt}(\omega)] - h \sum_{j=1}^{m} [y_{jt}(\omega) - \sum_{i=1}^{n} q_{ijt}(\omega)] \} d\omega \\
\text{s.t.} \quad & x_{i,t+1}(\omega) = \alpha[x_{it}(\omega) - \sum_{j=1}^{m} q_{ijt}(\omega)] + D_{it}(\omega), \text{ for all } i \in \mathcal{D},\ \tau = t, t+1, \ldots, T-1 \text{ and } \omega \in \Omega, \\
& y_{j,t+1}(\omega) = \beta[y_{jt}(\omega) - \sum_{i=1}^{n} q_{ijt}(\omega)] + S_{jt}(\omega), \text{ for all } j \in \mathcal{S},\ \tau = t, t+1, \ldots, T-1 \text{ and } \omega \in \Omega, \\
& \sum_{i=1}^{n} q_{ijt}(\omega) \le y_{jt}(\omega) \text{ for all } j \in \mathcal{S},\ \tau = t, t+1, \ldots, T \text{ and } \omega \in \Omega, \\
& \sum_{j=1}^{m} q_{ijt}(\omega) \le x_{it}(\omega) \text{ for all } i \in \mathcal{D},\ \tau = t, t+1, \ldots, T \text{ and } \omega \in \Omega, \\
& q_{ijt}(\omega) \ge 0 \text{ for all } i \in \mathcal{D},\ j \in \mathcal{S},\ \tau = t, t+1 \ldots, T \text{ and } \omega \in \Omega. \\
& x_{i\tau} = x_i, \quad y_{j\tau} = y_j, \quad \text{for } i \in \mathcal{D},\ j \in \mathcal{S},\ \omega \in \Omega.
\end{aligned}
$$

We denote by $Q^*_{ijt}(\omega)$, $\omega \in \Omega$, the optimal matching strategy, and by $x^*_{it}(\omega)$ and $y^*_{jt}(\omega)$ the associated state trajectory. Let $\bar{x}_{it}$, $\bar{y}_{jt}$ and $\bar{q}_{ijt}$ be the expectation of $x^*_{it}(\omega)$, $y^*_{jt}(\omega)$ and $q^*_{ijt}(\omega)$ over Ω, respectively. Because all sample paths satisfy the constraints of problem $(\mathrm{S}_\tau^{\mathbf{x},\mathbf{y}})$, as expectations, $(\bar{q}_{ijt}, \bar{x}_{it}, \bar{y}_{jt})$ is *feasible* for the deterministic problem $(\mathrm{P}_\tau^{\mathbf{x},\mathbf{y}})$, with the corresponding objective value equal to the optimal value of the stochastic problem $(\mathrm{S}_\tau^{\mathbf{x},\mathbf{y}})$. Therefore, the deterministic problem $(\mathrm{P}_\tau^{\mathbf{x},\mathbf{y}})$ has a larger optimal value than the stochastic problem $(\mathrm{S}_\tau^{\mathbf{x},\mathbf{y}})$. $\square$

Next we show that the heuristic policy suggested by the deterministic problem is asymptotically optimal. Consider a series of stochastic systems indexed by $k = 1, 2, \ldots$, with 1 representing the original system. We scale the time in system k so that the arrival of demand and supply in system k is k times more intense compared with the original system (equivalently, the clock is k times faster than the original system) in any given period t. Thus, instead of having random demand D_{it} for type $i \in \mathcal{D}$ and random supply S_{jt} for type $j \in \mathcal{S}$ arriving in a period, system k will have an amount of $D_{it}(k) = \sum_{\ell=1}^{k} D^\ell_{it}$ for type $i \in \mathcal{D}$ and $S_{jt}(k) = \sum_{\ell=1}^{k} S^\ell_{jt}$ for type $j \in \mathcal{S}$ arriving in period t, where the D^ℓ_{it}'s are i.i.d. with the same distribution as D_{it}, and the S^ℓ_{jt}'s are i.i.d. with the same distribution as S_{jt}. Let $V^k_t(\mathbf{x},\mathbf{y})$ be the value function in system k, $V^{\det(k)}_t(\mathbf{x},\mathbf{y})$ the value function for the deterministic model, and $V^{\mathrm{resolve}(k)}_t(\mathbf{x},\mathbf{y})$ the value for applying the deterministic resolving heuristic in system k. For simplicity, we write $V_t^{\text{det}}(\mathbf{x}, \mathbf{y}) = V_t^{\text{det}(1)}(\mathbf{x}, \mathbf{y})$ and $V_t^{\text{resolve}}(\mathbf{x}, \mathbf{y}) = V_t^{\text{resolve}(1)}(\mathbf{x}, \mathbf{y})$. The following theorem shows that the deterministic resolving heuristic is asymptotically optimal with a convergence rate $O(1/\sqrt{k})$.

Theorem 3.3 (Asymptotic optimality of the deterministic heuristic and rate of convergence) *In the stochastic system k, the deterministic resolving heuristic leads to the relative error* $[V_t^k(\mathbf{x}, \mathbf{y}) - V_t^{resolve(k)}]/V_t^k(\mathbf{x}, \mathbf{y}) = O(1/\sqrt{k})$ *as* $k \to \infty$.

3.6 Numerical Experiments

We now test the effectiveness of the deterministic heuristic. Consider a 10-period dynamic matching problem with 5 supply types and 5 demand types. For each instance of the problem, we generate the parameters uniformly at random as follows.

Let $r_{ij} \sim \text{Uniform}[50, 150]$ (for all $i \in \mathcal{D}$ and $j \in \mathcal{S}$), $c \sim \text{Uniform}[0, 50]$, $h \sim \text{Uniform}[0, 50]$, $\alpha \sim \text{Uniform}[0, 1]$, $\beta \sim \text{Uniform}[0, 1]$, $\mu = ES \sim \text{Uniform}[10, 25]$, $\lambda = ED \sim \text{Uniform}[10, 25]$, $\gamma \sim \text{Uniform}[0.8, 1]$.

In addition, we also randomly generate the initial state $(\mathbf{x}^0, \mathbf{y}^0)$ at the beginning of the first period. We let $x_i^0 \sim \text{Uniform}[0, 30]$ and $y_j^0 \sim \text{Uniform}[0, 30]$ for all $i \in \mathcal{D}$ and $j \in \mathcal{S}$.

We run two sets of numerical experiments as described as follows.

(a) Demand and supply follow a uniform distribution. For given realizations of λ_i and μ_j, we generate $\delta_i^d \sim \text{Uniform}[0, \lambda_i]$ and $\delta_j^s \sim \text{Uniform}[0, \mu_j]$. Then, we let $D_i \sim \text{Uniform}[\lambda_i - \delta_i^d, \lambda_i + \delta_i^d]$ and $S_j \sim \text{Uniform}[\mu_j - \delta_j^s, \mu_j + \delta_j^s]$.

(b) Demand and supply follow a normal distribution. For given realizations of λ_i and μ_j, we generate $\sigma_i^d \sim \text{Uniform}[0, \lambda_i/3]$ and $\sigma_j^s \sim \text{Uniform}[0, \mu_j/3]$. Then, we let $D_i \sim \text{Normal}(\lambda_i, \sigma_i^d)$ and $S_j \sim \text{Normal}(\mu_j, \sigma_j^s)$.

Note that all the parameters are generated independently. For each randomly generated instance, we solve the 10-period deterministic problem (P) and obtain the optimal value V^{det}, which is an upper bound of the optimal value V^{opt} of the stochastic problem. Let $\tilde{V}$ be the optimal value of the expected total discounted reward minus costs, when the deterministic heuristic is applied throughout the decision horizon. We calculate $\tilde{V}$ approximately by simulation: For each randomly generated sample path ω, in period t $(t = 1, \ldots, T)$ with state $(\mathbf{x}_t(\omega), \mathbf{y}_t(\omega))$, apply the optimal decision from solving the $(T - t + 1)$-period problem with initial state $(\mathbf{x}_t(\omega), \mathbf{y}_t(\omega))$; The total reward minus cost for the sample path ω can be easily calculated; Then we average over 5000 sample paths to obtain the approximate value of $\tilde{V}$. Since $(V^{\text{opt}} - \tilde{V})/V^{\text{opt}} \leq (V^{\text{det}} - \tilde{V})/V^{\text{det}}$, the relative error by the deterministic heuristic is small as long as the right-hand-side of the above inequality is small. Thus, we focus on $\rho = (V^{\text{det}} - \tilde{V})/V^{\text{det}}$ to measure the relative error.

For set (a) of the experiments, 600 instances are generated. Among the 600 instances, the maximum value of ρ is 21.24%, the mean is 9.84% and the median is 9.51%.

For set (b) of the experiments, 820 instances are generated. Among the 820 instances, the maximum value of ρ is 19.24%, the mean is 7.24% and the median is 6.79%.

The empirical cumulative distribution functions for the two sets of ρ values are shown in Figures 3.5 and 3.6, respectively. We see that the values of ρ are relatively small. Since ρ is just an upper bound of the relative error, the relative error would be even smaller.

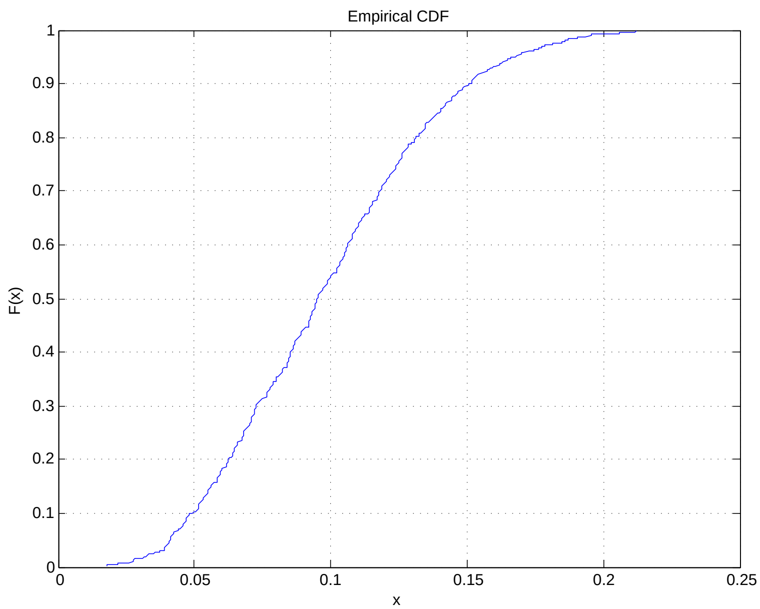


Figure 3.5: Empirical cdf of ρ: Uniformly distributed demand and supply

3.7 Conclusion

In this chapter, we generalize, the Monge sequence condition that is necessary and sufficient for a greedy algorithm to solve a deterministic and balanced transportation problem, to a stochastic and dynamic matching problem. The generalization involves extending the notion of "augmenting path" to a stochastic and dynamic setting through backward induction. The modified Monge conditions on the reward matrix that we discover are sufficient, and in a robust sense, necessary, to guarantee a priority structure (somewhat weaker than the greedy algorithm) in the optimal matching policy for the general problem with intertemporally random demand and supply. We also propose to solve the deterministic counterpart for providing heuristic solutions for the stochastic problem. We show that the deterministic heuristic provides an upper bound on the performance of the stochastic problem, and is asymptotically optimal as we scale up demand and supply.

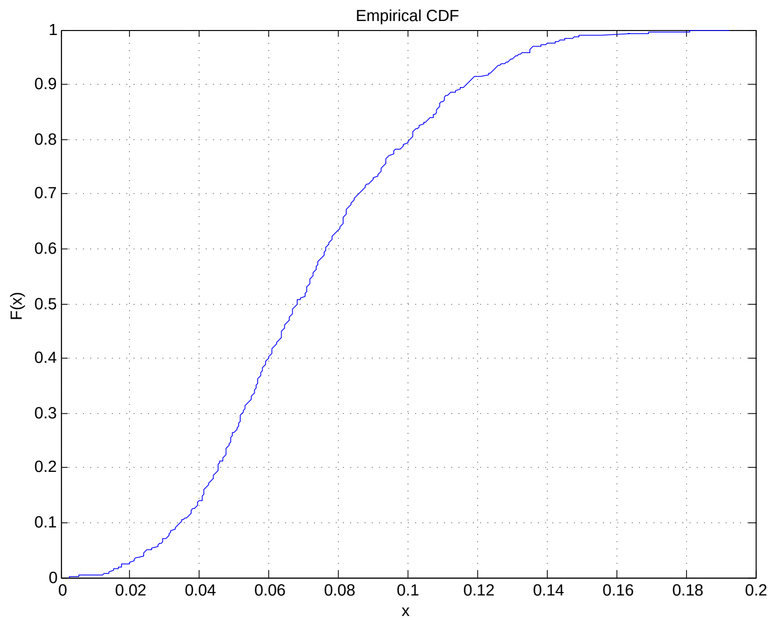


Figure 3.6: Empirical cdf of ρ: Normally distributed demand and supply

3.8 Appendix

Proof of Lemma 3.1. Let $\mathbf{u} = \mathbf{x} - \mathbf{1}^m\mathbf{Q}^\mathsf{T}$ and $\mathbf{v} = \mathbf{y} - \mathbf{1}^n\mathbf{Q}$. We have

$$\begin{aligned}
&H_t(\mathbf{Q} + \epsilon \mathbf{e}_{ij}^{n\times m} - \epsilon \mathbf{e}_{i'j'}^{n\times m}, \mathbf{x}, \mathbf{y}) - H_t(\mathbf{Q}, \mathbf{x}, \mathbf{y}) \\
=&(r_{ij} - r_{i'j'})\epsilon + \gamma E V_{t+1}(\alpha\mathbf{u} + \mathbf{D} - \alpha\epsilon\mathbf{e}_i^n + \alpha\epsilon\mathbf{e}_{i'}^n, \beta\mathbf{v} + \mathbf{S} - \beta\epsilon\mathbf{e}_j^m + \beta\epsilon\mathbf{e}_{j'}^m) - \gamma E V_{t+1}(\alpha\mathbf{u} + \mathbf{D}, \beta\mathbf{v} + \mathbf{S}) \\
\geq&(r_{ij} - r_{i'j'})\epsilon + \gamma \max\{\alpha, \beta\}\,\epsilon(r_{i'j'} - r_{ij}) \\
=&(1 - \gamma\max\{\alpha, \beta\})\epsilon(r_{ij} - r_{i'j'}) \geq 0,
\end{aligned}$$

where the first inequality is due to Lemma 3.3 and the last inequality is due to that $\alpha, \beta, \gamma \leq 1$ and $(i, j) \succeq (i', j')$. $\square$

Proof of Lemma 3.2. The result holds trivially for $t = T+1$ by the boundary condition $V_{T+1}(\mathbf{x}, \mathbf{y}) \equiv 0$. Suppose it holds for $t+1$. We show that it also holds for t.

Consider a given $(\mathbf{x}, \mathbf{y})$ with $x_i > 0$ and $y_j > 0$, $\epsilon_t^1 \in [0, x_i]$ and $\epsilon_t^2 \in [0, y_j]$. Let

$$\hat{\mathbf{Q}} \in \underset{\mathbf{Q} \in \{\mathbf{Q} \geq \mathbf{0} | \mathbf{u} \geq \mathbf{0}, \mathbf{v} \geq \mathbf{0}\}}{\arg\max} H_t(\mathbf{Q}, \mathbf{x}, \mathbf{y}).$$

We claim that:

CLAIM. There exist nonnegative numbers $\eta_{j''}^t$ for $j'' \in \mathcal{S}$ and $\xi_{i''}^t$ for $i'' \in \mathcal{D}$ such that $\sum_{j''\in\mathcal{S}} \eta_{j''}^t \leq \epsilon_t^1$, $\sum_{i''\in\mathcal{D}} \xi_{i''}^t \leq \epsilon_t^2$, and the decision $\widetilde{\mathbf{Q}} = \hat{\mathbf{Q}} + \sum_{i''\in\mathcal{D}}(\xi_{i''}^t \mathbf{e}_{i''j'}^{n\times m} - \xi_{i''}^t \mathbf{e}_{i''j}^{n\times m}) + \sum_{j''\in\mathcal{S}}(\eta_{j''}^t \mathbf{e}_{i'j''}^{n\times m} - \eta_{j''}^t \mathbf{e}_{ij''}^{n\times m})$ is feasible under the state $(\mathbf{x} - \epsilon_t^1\mathbf{e}_i^n + \epsilon_t^1\mathbf{e}_{i'}^n, \mathbf{y} - \epsilon_t^2\mathbf{e}_j^m + \epsilon_t^2\mathbf{e}_{j'}^m)$.

Proof of Claim. We construct $\eta^t_{j''}$ as follows. Let $\eta^t_1 = \min\{\hat{q}_{i1}, \epsilon^1_t\}$. Then, recursively, let $\eta^t_{j''} = \min\left\{\hat{q}_{ij''}, \epsilon^1_t - \sum_{k=1}^{j''-1} \eta^t_k\right\}$ for $j'' = 2, \ldots, m$.

We first prove $\epsilon^1_t - \sum_{k=1}^{j''} \eta^t_k = (\epsilon^1_t - \sum_{k=1}^{j''} \hat{q}_{ik})^+$ for all $j'' \in \mathcal{S}$ by induction, which guarantees that $\eta^t_{j''} \geq 0$ and $\sum_{j''=1}^m \eta^t_{j''} \leq \epsilon^1_t$. For $j = 1$, $\epsilon^1_t - \eta^t_1 = \epsilon^1_t - \min\{\hat{q}_{i1}, \epsilon^1_t\} = (\epsilon^1_t - \hat{q}_{i1})^+$. Thus the equation holds for $j'' = 1$. Suppose it holds for j''. Then for $j'' + 1$,

$$\begin{aligned}
\epsilon^1_t - \sum_{k=1}^{j''+1} \eta^t_k =&(\epsilon^1_t - \sum_{k=1}^{j''} \eta^t_k) - \eta^t_{j''+1} = (\epsilon^1_t - \sum_{k=1}^{j''} \eta^t_k) - \min\{\epsilon^1_t - \sum_{k=1}^{j''} \eta^t_k, \hat{q}_{i,j''+1}\} \\
=&(\epsilon^1_t - \sum_{k=1}^{j''} \hat{q}_{ik})^+ - \min\{(\epsilon^1_t - \sum_{k=1}^{j''} \hat{q}_{ik})^+, \hat{q}_{i,j''+1}\} \\
=&[(\epsilon^1_t - \sum_{k=1}^{j''} \hat{q}_{ik})^+ - \hat{q}_{i,j''+1}]^+ = [\epsilon^1_t - \sum_{k=1}^{j''} \hat{q}_{ik} - \hat{q}_{i,j''+1}]^+ = [\epsilon^1_t - \sum_{k=1}^{j''+1} \hat{q}_{ik}]^+,
\end{aligned}$$

which completes the induction.

Hence, we have $\epsilon^1_t - \sum_{j''=1}^m \eta^t_{j''} = (\epsilon^1_t - \sum_{j''=1}^m \hat{q}_{ij''})^+$. Case (i): If $\sum_{j''=1}^m \hat{q}_{ij''} < \epsilon^1_t$, then $\epsilon^1_t - \sum_{j''=1}^m \eta^t_{j''} = \epsilon^1_t - \sum_{j''=1}^m \hat{q}_{ij''}$, implying that $\sum_{j''=1}^m \hat{q}_{ij''} = \sum_{j''=1}^m \eta^t_{j''}$. Case (ii): If $\sum_{j''=1}^m \hat{q}_{ij''} \geq \epsilon^1_t$, then $\epsilon^1_t - \sum_{j''=1}^m \eta^t_{j''} = 0$, implying that $\epsilon^1_t = \sum_{j''=1}^m \eta^t_{j''}$. Combining the two cases, we have $\sum_{j''=1}^m \eta^t_{j''} \leq \epsilon^1_t$.

Now we show that the decision $\bar{\mathbf{Q}} = \hat{\mathbf{Q}} + \sum_{j''\in\mathcal{S}}(\eta^t_{j''}\mathbf{e}^{n\times m}_{i'j''} - \eta^t_{j''}\mathbf{e}^{n\times m}_{ij''})$ is feasible under the state $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \equiv (\mathbf{x} - \epsilon^1_t\mathbf{e}^n_i + \epsilon^1_t\mathbf{e}^n_{i'}, \mathbf{y})$ in period t.

Since $\hat{\mathbf{Q}}$ is optimal for the state $(\mathbf{x}, \mathbf{y})$, a fortiori, $\hat{\mathbf{Q}}$ is feasible, i.e., $\hat{\mathbf{Q}} \geq 0$, $\mathbf{1}^m\hat{\mathbf{Q}}^\intercal \leq \mathbf{x}$ and $\mathbf{1}^n\hat{\mathbf{Q}} \leq \mathbf{y}$. We show that $\bar{\mathbf{Q}}$ is feasible for the new state $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \geq \mathbf{0}$, where the latter inequality is due to $x_i > 0$ and $\epsilon^1_t \in [0, x_i]$. To this end, it suffices to show that $\bar{\mathbf{Q}} \geq 0$, $\mathbf{1}^m\bar{\mathbf{Q}}^\intercal \leq \bar{\mathbf{x}}$ and $\mathbf{1}^n\bar{\mathbf{Q}} \leq \bar{\mathbf{y}}$.

First, for all j, because $0 \leq \eta^t_j \leq \hat{q}_{ij}$, we have $\bar{q}_{ij} = \hat{q}_{ij} - \eta^t_j \geq 0$. Also, it is clear that $\bar{q}_{i'j} = \hat{q}_{i'j} + \eta^t_j \geq 0$ for all j. For any $i'' \neq i, i'$, we have $\bar{q}_{i''j} = \hat{q}_{i''j} \geq 0$ for all j. Thus, $\bar{\mathbf{Q}} \geq 0$.

Second, we have $\mathbf{1}^m\bar{\mathbf{Q}}^\intercal_i = \mathbf{1}^m\hat{\mathbf{Q}}^\intercal_i - \sum_{j''\in\mathcal{S}} \eta^t_{j''} = \sum_{j'=1}^m \hat{q}_{ij'} - \sum_{j''\in\mathcal{S}} \eta^t_{j''}$. If $\sum_{j=1}^m \hat{q}_{ij} < \epsilon^1_t$, we have $\mathbf{1}^m\bar{\mathbf{Q}}^\intercal_i = \sum_{j'=1}^m \hat{q}_{ij'} - \sum_{j''\in\mathcal{S}} \eta^t_{j''} = 0 \leq \bar{x}_i$. If $\sum_{j=1}^m \hat{q}_{ij} \geq \epsilon^1_t$, then $\epsilon^1_t = \sum_{j''\in\mathcal{S}} \eta^t_{j''}$. Thus, $\mathbf{1}^m\bar{\mathbf{Q}}^\intercal_i = \mathbf{1}^m\hat{\mathbf{Q}}^\intercal_i - \sum_{j''\in\mathcal{S}} \eta^t_{j''} = \mathbf{1}^m\hat{\mathbf{Q}}^\intercal_i - \epsilon^1_t \leq x_i - \epsilon^1_t = \bar{x}_i$. We also have $\mathbf{1}^m\bar{\mathbf{Q}}^\intercal_{i'} = \mathbf{1}^m\hat{\mathbf{Q}}^\intercal_{i'} + \sum_{j''\in\mathcal{S}} \eta^t_{j''} \leq \mathbf{1}^m\hat{\mathbf{Q}}^\intercal_{i'} + \epsilon^1_t \leq x_{i'} + \epsilon^1_t = \bar{x}_{i'}$. For any $i'' \neq i, i'$, $\mathbf{1}^m\bar{\mathbf{Q}}^\intercal_{i''} = \mathbf{1}^m\hat{\mathbf{Q}}^\intercal_{i''} \leq x_{i''} = \bar{x}_{i''}$. Therefore, $\mathbf{1}^m\bar{\mathbf{Q}}^\intercal \leq \bar{\mathbf{x}}$.

Finally, $\mathbf{1}^n\bar{\mathbf{Q}} = \mathbf{1}^n\hat{\mathbf{Q}} + \mathbf{1}^n(-\sum_{j'=1}^m \eta^t_{j'}\mathbf{e}^{n\times m}_{ij'} + \sum_{j'=1}^m \eta^t_{j'}\mathbf{e}^{n\times m}_{i'j'}) = \mathbf{1}^n\hat{\mathbf{Q}} + 0 \leq \mathbf{y} = \bar{\mathbf{y}}$.

Define $\xi^t_1 = \min\{\bar{q}_{1j}, \epsilon^2_t\}$ and $\xi^t_{i''} = \min\left\{\bar{q}_{i''j}, \epsilon^2_t - \sum_{k=1}^{i''-1} \xi^t_k\right\}$ for $i'' = 2, \ldots, n$. Following a symmetric analysis, we can show that the decision $\widetilde{\mathbf{Q}} = \bar{\mathbf{Q}} + \sum_{i''\in\mathcal{D}}(\xi^t_{i''}\mathbf{e}^{n\times m}_{i''j'} - \xi^t_{i''}\mathbf{e}^{n\times m}_{i''j})$ is feasible under the state $(\mathbf{x} - \epsilon^1_t\mathbf{e}^n_i + \epsilon^1_t\mathbf{e}^n_{i'}, \mathbf{y} - \epsilon^2_t\mathbf{e}^m_j + \epsilon^2_t\mathbf{e}^m_{j'}) = (\bar{\mathbf{x}}, \bar{\mathbf{y}} - \epsilon^2_t\mathbf{e}^m_j + \epsilon^2_t\mathbf{e}^m_{j'})$. This proves the claim. □

Now denote by $\mathbf{u}$ and $\mathbf{v}$ the post-matching levels under the state $(\mathbf{x}, \mathbf{y})$ and the decision $\hat{\mathbf{Q}}$ in period t. Define $\epsilon^1_{t+1} = \alpha(\epsilon^1_t - \sum_{j''\in\mathcal{S}} \eta^t_{j''})$ and $\epsilon^2_{t+1} = \beta(\epsilon^2_t - \sum_{i''\in\mathcal{D}} \xi^t_{i''})$. We have:

$$\begin{aligned}
&V_t(\mathbf{x} - \epsilon^1_t\mathbf{e}^n_i + \epsilon^1_t\mathbf{e}^n_{i'}, \mathbf{y} - \epsilon^2_t\mathbf{e}^m_j + \epsilon^2_t\mathbf{e}^m_{j'}) - V_t(\mathbf{x}, \mathbf{y}) \\
\geq &H_t(\widetilde{\mathbf{Q}}, \mathbf{x} - \epsilon^1_t\mathbf{e}^n_i + \epsilon^1_t\mathbf{e}^n_{i'}, \mathbf{y} - \epsilon^2_t\mathbf{e}^m_j + \epsilon^2_t\mathbf{e}^m_{j'}) - H_t(\hat{\mathbf{Q}}, \mathbf{x}, \mathbf{y})
\end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{i'' \in \mathcal{D}} \xi_{i''}^t (r_{i''j'} - r_{i''j}) + \sum_{j'' \in \mathcal{S}} \eta_{j''}^t (r_{i'j''} - r_{ij''}) \\
&\quad + \gamma E V_{t+1}(\alpha[\mathbf{u} - (\epsilon_t^1 - \sum_{j'' \in \mathcal{S}} \eta_{j''}^t)\mathbf{e}_i^n + (\epsilon_t^1 - \sum_{j'' \in \mathcal{S}} \eta_{j''}^t)\mathbf{e}_{i'}^n] + \mathbf{D}, \beta[\mathbf{v} - (\epsilon_t^2 - \sum_{i'' \in \mathcal{D}} \xi_{i''}^t)\mathbf{e}_j^m + (\epsilon_t^2 - \sum_{i'' \in \mathcal{D}} \xi_{i''}^t)\mathbf{e}_{j'}^m] + \mathbf{S}) \\
&\quad - \gamma E V_{t+1}(\alpha \mathbf{u} + \mathbf{D}, \beta \mathbf{v} + \mathbf{S}) \\
&= \sum_{i'' \in \mathcal{D}} \xi_{i''}^t (r_{i''j'} - r_{i''j}) + \sum_{j'' \in \mathcal{S}} \eta_{j''}^t (r_{i'j''} - r_{ij''}) \\
&\quad + \gamma E V_{t+1}(\alpha \mathbf{u} - \epsilon_{t+1}^1 \mathbf{e}_i^n + \epsilon_{t+1}^1 \mathbf{e}_{i'}^n + \mathbf{D}, \beta \mathbf{v} - \epsilon_{t+1}^2 \mathbf{e}_j^m + \epsilon_{t+1}^2 \mathbf{e}_{j'}^m + \mathbf{S}) - \gamma E V_{t+1}(\alpha \mathbf{u} + \mathbf{D}, \beta \mathbf{v} + \mathbf{S}).
\end{aligned} \tag{3.3}
$$

Let $\mathbf{X}_{t+1} = \alpha\mathbf{u} + \mathbf{D}$ and $\mathbf{Y}_{t+1} = \beta\mathbf{v} + \mathbf{S}$. By the induction hypothesis, there exist $K_{j''}^\tau$ and $L_{i''}^\tau$ for $j'' \in \mathcal{S}$, $i'' \in \mathcal{D}$ and $\tau = t+1, \ldots, T$ such that $\sum_{\tau=t+1}^T \sum_{j'' \in \mathcal{S}} K_{j''}^\tau \leq \epsilon_{t+1}^1$, $\sum_{\tau=t+1}^T \sum_{i'' \in \mathcal{D}} L_{i''}^\tau \leq \epsilon_{t+1}^2$ and

$$
\begin{aligned}
&V_{t+1}(\mathbf{X}_{t+1} - \epsilon_{t+1}^1 \mathbf{e}_i^n + \epsilon_{t+1}^1 \mathbf{e}_{i'}^n, \mathbf{Y}_{t+1} - \epsilon_{t+1}^2 \mathbf{e}_j^m + \epsilon_{t+1}^2 \mathbf{e}_{j'}^m) - V_{t+1}(\mathbf{X}_{t+1}, \mathbf{Y}_{t+1}) \\
&\geq \sum_{\tau=t+1}^T \gamma^{\tau-t-1} [\sum_{j'' \in \mathcal{S}} K_{j''}^\tau (r_{i'j''} - r_{ij''}) + \sum_{i'' \in \mathcal{D}} L_{i''}^\tau (r_{i''j'} - r_{i''j})].
\end{aligned} \tag{3.4}
$$

Let $\eta_{j''}^\tau = EK_{j''}^\tau$ and $\xi_{i''}^\tau = EL_{i''}^\tau$ for $j'' \in \mathcal{S}$, $i'' \in \mathcal{D}$ and $\tau = t+1, \ldots, T$. We have

$$
\begin{aligned}
&V_t(\mathbf{x} - \epsilon_t^1 \mathbf{e}_i^n + \epsilon_t^1 \mathbf{e}_{i'}^n, \mathbf{y} - \epsilon_t^2 \mathbf{e}_j^m + \epsilon_t^2 \mathbf{e}_{j'}^m) - V_t(\mathbf{x}, \mathbf{y}) \\
&\geq \sum_{i'' \in \mathcal{D}} \xi_{i''}^t (r_{i''j'} - r_{i''j}) + \sum_{j'' \in \mathcal{S}} \eta_{j''}^t (r_{i'j''} - r_{ij''}) \\
&\quad + \gamma E V_{t+1}(\alpha \mathbf{u} - \epsilon_{t+1}^1 \mathbf{e}_i^n + \epsilon_{t+1}^1 \mathbf{e}_{i'}^n + \mathbf{D}, \beta \mathbf{v} - \epsilon_{t+1}^2 \mathbf{e}_j^m + \epsilon_{t+1}^2 \mathbf{e}_{j'}^m + \mathbf{S}) - \gamma E V_{t+1}(\alpha \mathbf{u} + \mathbf{D}, \beta \mathbf{v} + \mathbf{S}) \\
&\geq \sum_{i'' \in \mathcal{D}} \xi_{i''}^t (r_{i''j'} - r_{i''j}) + \sum_{j'' \in \mathcal{S}} \eta_{j''}^t (r_{i'j''} - r_{ij''}) + \sum_{\tau=t+1}^T \gamma^{\tau-t} [\sum_{j'' \in \mathcal{S}} \eta_{j''}^\tau (r_{i'j''} - r_{ij''}) + \sum_{i'' \in \mathcal{D}} \xi_{i''}^\tau (r_{i''j'} - r_{i''j})] \\
&= \sum_{\tau=t}^T \gamma^{\tau-t} [\sum_{j'' \in \mathcal{S}} \eta_{j''}^\tau (r_{i'j''} - r_{ij''}) + \sum_{i'' \in \mathcal{D}} \xi_{i''}^\tau (r_{i''j'} - r_{i''j})],
\end{aligned}
$$

where the first inequality is (3.3) and the second inequality is due to (3.4).

Moreover, $\sum_{\tau=t}^T \sum_{j'' \in \mathcal{S}} \eta_{j''}^\tau = \sum_{j'' \in \mathcal{S}} \eta_{j''}^t + E \sum_{\tau=t+1}^T \sum_{j'' \in \mathcal{S}} K_{j''}^\tau \leq \sum_{j'' \in \mathcal{S}} \eta_{j''}^t + \epsilon_{t+1}^1 = \sum_{j'' \in \mathcal{S}} \eta_{j''}^t + \alpha(\epsilon_t^1 - \sum_{j'' \in \mathcal{S}} \eta_{j''}^t) = \alpha\epsilon_t^1 + (1-\alpha) \sum_{j'' \in \mathcal{S}} \eta_{j''}^t \leq \epsilon_t^1$, because $\sum_{j'' \in \mathcal{S}} \eta_{j''}^t \leq \epsilon_t^1$. Similarly, $\sum_{\tau=t}^T \sum_{i'' \in \mathcal{D}} \xi_{i''}^\tau = \sum_{i'' \in \mathcal{D}} \xi_{i''}^t + E \sum_{\tau=t+1}^T \sum_{i'' \in \mathcal{D}} L_{i''}^\tau = \sum_{i'' \in \mathcal{D}} \xi_{i''}^t + \epsilon_{t+1}^2 \leq \sum_{i'' \in \mathcal{D}} \xi_{i''}^t + \beta(\epsilon_t^2 - \sum_{i'' \in \mathcal{D}} \xi_{i''}^t) \leq \epsilon_t^2$. This completes the induction. □

Proof of Lemma 3.3. By Lemma 3.2, there exist nonnegative numbers $\eta_{j''}^\tau$ and $\xi_{i''}^\tau$ for $j'' \in \mathcal{S}$, $i'' \in \mathcal{D}$ and $\tau = t, \ldots, T+1$ such that $\sum_{\tau=t}^T \sum_{j'' \in \mathcal{S}} \eta_{j''}^\tau \leq \epsilon_1$, $\sum_{\tau=t}^T \sum_{i'' \in \mathcal{D}} \xi_{i''}^\tau \leq \epsilon_2$ and

$$
V_t(\mathbf{x} - \epsilon_1 \mathbf{e}_i^n + \epsilon_1 \mathbf{e}_{i'}^n, \mathbf{y} - \epsilon_2 \mathbf{e}_j^m + \epsilon_2 \mathbf{e}_{j'}^m) - V_t(\mathbf{x}, \mathbf{y})
$$

$$\geq \sum_{\tau=t}^{T} \gamma^{\tau-t} [\sum_{j'' \in \mathcal{S}} \eta_{j''}^{\tau} (r_{i'j''} - r_{ij''}) + \sum_{i'' \in \mathcal{D}} \xi_{i''}^{\tau} (r_{i''j'} - r_{i''j})]. \tag{3.5}$$

Since $(i, j) \succeq (i', j')$, there exists a decreasing sequence connecting the two arcs. Without loss of generality, we choose a path in the form of $(i, j) = (i_1, j_1) \succeq (i_1, j_2) \succeq (i_2, j_2) \succeq \cdots \succeq (i_\ell, j_\ell) = (i', j')$, and the proof for the other forms would be analogous. The condition $(i_k, j_k) \succeq (i_{k+1}, j_k)$ implies that $r_{i_k j_k} + r_{i_{k+1} j''} \geq r_{i_k j''} + r_{i_{k+1} j_k}$, i.e., $r_{i_{k+1} j''} - r_{i_k j''} \geq r_{i_{k+1} j_k} - r_{i_k j_k}$. Thus

$$r_{i'j''} - r_{ij''} = r_{i_\ell j''} - r_{i_1 j''} = \sum_{k=1}^{\ell-1} (r_{i_{k+1} j''} - r_{i_k j''}) \geq \sum_{k=1}^{\ell-1} (r_{i_{k+1} j_k} - r_{i_k j_k}). \tag{3.6}$$

Likewise, the condition $(i_{k+1}, j_k) \succeq (i_{k+1}, j_{k+1})$ implies that

$$r_{i''j'} - r_{i''j} = r_{i'' j_\ell} - r_{i'' j_1} = \sum_{k=1}^{\ell-1} (r_{i'' j_{k+1}} - r_{i'' j_k}) \geq \sum_{k=1}^{\ell-1} (r_{i_{k+1} j_{k+1}} - r_{i_{k+1} j_k}). \tag{3.7}$$

Then,

$$\begin{aligned}
& V_t(\mathbf{x} - \epsilon \mathbf{e}_i^n + \epsilon \mathbf{e}_{i'}^n, \mathbf{y} - \epsilon \mathbf{e}_j^m + \epsilon \mathbf{e}_{j'}^m) - V_t(\mathbf{x}, \mathbf{y}) \\
\geq & \sum_{\tau=t}^{T} \gamma^{\tau-t} \sum_{j'' \in \mathcal{S}} \eta_{j''}^{\tau} \sum_{k=1}^{\ell-1} (r_{i_{k+1} j_k} - r_{i_k j_k}) + \sum_{\tau=t}^{T} \gamma^{\tau-t} \sum_{i'' \in \mathcal{D}} \xi_{i''}^{\tau} \sum_{k=1}^{\ell-1} (r_{i_{k+1} j_{k+1}} - r_{i_{k+1} j_k}) \\
= & (\sum_{\tau=t}^{T} \gamma^{\tau-t} \sum_{j'' \in \mathcal{S}} \eta_{j''}^{\tau} - \sum_{\tau=t}^{T} \gamma^{\tau-t} \sum_{i'' \in \mathcal{D}} \xi_{i''}^{\tau}) \sum_{k=1}^{\ell-1} (r_{i_{k+1} j_k} - r_{i_k j_k}) \\
& + \sum_{\tau=t}^{T} \gamma^{\tau-t} \sum_{i'' \in \mathcal{D}} \xi_{i''}^{\tau} [\sum_{k=1}^{\ell-1} (r_{i_{k+1} j_k} - r_{i_k j_k}) + \sum_{k=1}^{\ell-1} (r_{i_{k+1} j_{k+1}} - r_{i_{k+1} j_k})] \\
= & (\sum_{\tau=t}^{T} \gamma^{\tau-t} \sum_{j'' \in \mathcal{S}} \eta_{j''}^{\tau} - \sum_{\tau=t}^{T} \gamma^{\tau-t} \sum_{i'' \in \mathcal{D}} \xi_{i''}^{\tau}) \sum_{k=1}^{\ell-1} (r_{i_{k+1} j_k} - r_{i_k j_k}) + \sum_{\tau=t}^{T} \gamma^{\tau-t} \sum_{i'' \in \mathcal{D}} \xi_{i''}^{\tau} (r_{i_\ell j_\ell} - r_{i_1 j_1}),
\end{aligned} \tag{3.8}$$

where the first inequality is due to (3.5), (3.6) and (3.7). Moreover,

$$\sum_{k=1}^{\ell-1} (r_{i_{k+1} j_k} - r_{i_k j_k}) = r_{i_\ell j_\ell} - r_{i_1 j_1} + \sum_{k=1}^{\ell-1} (r_{i_{k+1} j_k} - r_{i_{k+1} j_{k+1}}) \geq r_{i_\ell j_\ell} - r_{i_1 j_1}, \tag{3.9}$$

where the inequality follows from the condition $(i_{k+1}, j_k) \succeq (i_{k+1}, j_{k+1})$ that implies $r_{i_{k+1} j_k} - r_{i_{k+1} j_{k+1}} \geq 0$. Without loss of generality, let us assume $\sum_{\tau=t}^{T} \gamma^{\tau-t} \sum_{j'' \in \mathcal{S}} \eta_{j''}^{\tau} \geq \sum_{\tau=t}^{T} \gamma^{\tau-t} \sum_{i'' \in \mathcal{D}} \xi_{i''}^{\tau}$. Then,

$$\begin{aligned}
& V_t(\mathbf{x} - \epsilon_1 \mathbf{e}_i^n + \epsilon_1 \mathbf{e}_{i'}^n, \mathbf{y} - \epsilon_2 \mathbf{e}_j^m + \epsilon_2 \mathbf{e}_{j'}^m) - V_t(\mathbf{x}, \mathbf{y}) \\
& \geq \sum_{\tau=t}^{T} \gamma^{\tau-t} \sum_{j'' \in \mathcal{S}} \eta_{j''}^{\tau} (r_{i_\ell j_\ell} - r_{i_1 j_1}) \geq \sum_{\tau=t}^{T} \sum_{j'' \in \mathcal{S}} \eta_{j''}^{\tau} (r_{i_\ell j_\ell} - r_{i_1 j_1}) \\
& \geq \epsilon_1 (r_{i_\ell j_\ell} - r_{i_1 j_1}) \geq \epsilon (r_{i_\ell j_\ell} - r_{i_1 j_1}) = \epsilon (r_{i'j'} - r_{ij}),
\end{aligned}$$

where the first inequality is due to (3.8), (3.9) and the assumption that $\sum_{\tau=t}^{T}\gamma^{\tau-t}\sum_{j''\in\mathcal{S}}\eta_{j''}^{\tau} \geq \sum_{\tau=t}^{T}\gamma^{\tau-t}\sum_{i''\in\mathcal{D}}\xi_{i''}^{\tau}$, and the remaining inequalities are due to $r_{i_\ell j_\ell} = r_{i'j'} \leq r_{ij} = r_{i_1 j_1}$ implied by $(i,j) \succeq (i',j')$. The first inequality can be shown similarly for the case $\sum_{\tau=t}^{T}\gamma^{\tau-t}\sum_{j''\in\mathcal{S}}\eta_{j''}^{\tau} < \sum_{\tau=t}^{T}\gamma^{\tau-t}\sum_{i''\in\mathcal{D}}\xi_{i''}^{\tau}$, with (3.8) rewritten in terms of $(\sum_{\tau=t}^{T}\gamma^{\tau-t}\sum_{i''\in\mathcal{D}}\xi_{i''}^{\tau} - \sum_{\tau=t}^{T}\gamma^{\tau-t}\sum_{j''\in\mathcal{S}}\eta_{j''}^{\tau})$. □

Proof of Theorem 3.2. We prove this theorem by induction on t. For $t = T+1$, it is obvious that the result holds. Suppose that the result holds for period $t+1$. We show that it also holds for period t.

Now consider period $t \leq T$. Without loss of generality, we can assume that both x_i and y_j are positive in period t. Otherwise if $x_i = 0$ or $y_j = 0$, the result clearly holds because the only feasible choice for q_{ij} is zero and thus $q_{ij}^* = 0 = \min\{x_i, y_j\}$.

Fix any $(\mathbf{x},\mathbf{y}) > \mathbf{0}$. Suppose that in the optimal matching policy $\mathbf{Q}^*$, $q_{ij}^* < \min\{x_i, y_j\}$. It is sufficient to show that there exists $\epsilon > 0$ such that an alternative matching plan $\bar{\mathbf{Q}}$, in which $\bar{q}_{ij} = q_{ij}^* + \epsilon$, weakly dominates $\mathbf{Q}^*$. In other words, the firm can improve weakly by matching ϵ more of type i demand and type j supply.

One of the following scenarios must hold for the post-matching quantities u_i^* and v_j^*: Case (i) $u_i^* > 0$ and $v_j^* > 0$; Case (ii) $u_i^* = 0$ and $v_j^* > 0$, or $u_i^* > 0$ and $v_j^* = 0$; Case (iii) $u_i^* = 0$ and $v_j^* = 0$. The ideas of constructing a weakly dominating policy for cases (i) and (iii) are representative, which will be repetitively used later to prove the global priority structure (see Theorem 3.1).

Case (i): $u_i^* > 0$ and $v_j^* > 0$. We choose $\epsilon > 0$ such that $u_i^* - \epsilon > 0$ and $v_j^* - \epsilon > 0$. Consider an alternative matching plan $\bar{\mathbf{Q}} = \mathbf{Q}^* + \epsilon\mathbf{e}_{ij}^{n\times m}$, which is clearly feasible. Then,

$$\begin{aligned}&H_t(\bar{\mathbf{Q}},\mathbf{x},\mathbf{y}) - H_t(\mathbf{Q}^*,\mathbf{x},\mathbf{y})\\ =&r_{ij}\epsilon + (h+c)\epsilon + \gamma EV_{t+1}(\alpha\mathbf{u}^* + \mathbf{D} - \alpha\epsilon\mathbf{e}_i^n, \beta\mathbf{v}^* + \mathbf{S} - \beta\epsilon\mathbf{e}_j^m) - \gamma EV_{t+1}(\alpha\mathbf{u}^* + \mathbf{D}, \beta\mathbf{v}^* + \mathbf{S}).\end{aligned} \tag{3.10}$$

If $t = T$, then $H_t(\bar{\mathbf{Q}},\mathbf{x},\mathbf{y}) - H_t(\mathbf{Q}^*,\mathbf{x},\mathbf{y}) = r_{ij}\epsilon + (h+c)\epsilon \geq 0$.

If $t < T$, then by the induction hypothesis, in period $t+1$, the optimal quantity to match between type i demand and type j supply is $q_{ij}^*(t+1) = \min\{x_i(t+1), y_j(t+1)\}$. Consider the case $\beta \geq \alpha$. It is easy to see that

$$V_{t+1}(\alpha\mathbf{u}^* + \mathbf{D} - \alpha\epsilon\mathbf{e}_i^n, \beta\mathbf{v}^* + \mathbf{S} - \beta\epsilon\mathbf{e}_j^m) = V_{t+1}(\alpha\mathbf{u}^* + \mathbf{D} + (\beta-\alpha)\epsilon\mathbf{e}_i^n, \beta\mathbf{v}^* + \mathbf{S}) - \beta\epsilon r_{ij}, \tag{3.11}$$

because of the greedy matching of pair (i,j) for the subsequent periods.

Now compare two systems that start in period $t+1$ with the states $(\alpha\mathbf{u}^* + \mathbf{D} + (\beta-\alpha)\epsilon\mathbf{e}_i^n, \beta\mathbf{v}^* + \mathbf{S})$ and $(\alpha\mathbf{u}^* + \mathbf{D}, \beta\mathbf{v}^* + \mathbf{S})$, respectively. The former system has the option of holding the additional amount $(\beta-\alpha)\epsilon$ of type i demand and mimicking the optimal matching policy of the latter system from period $t+1$ to period T. In this way, the former system incurs the extra cost $c(\beta-\alpha)\epsilon\sum_{\tau=0}^{T-t}\alpha^\tau\gamma^\tau \leq$

$c(\beta-\alpha)\epsilon/(1-\alpha\gamma) \leq c\epsilon$. That is,

$$V_{t+1}(\alpha\mathbf{u}^* + \mathbf{D} + (\beta-\alpha)\epsilon\mathbf{e}_i^n, \beta\mathbf{v}^* + \mathbf{S}) \geq V_{t+1}(\alpha\mathbf{u}^* + \mathbf{D}, \beta\mathbf{v}^* + \mathbf{S}) - c\epsilon. \tag{3.12}$$

Then, combining (3.10), (3.11) and (3.12), we have

$$\begin{aligned}
&H_t(\bar{\mathbf{Q}}, \mathbf{x}, \mathbf{y}) - H_t(\mathbf{Q}^*, \mathbf{x}, \mathbf{y}) \\
\geq& r_{ij}\epsilon + (h+c)\epsilon - \gamma\beta\epsilon r_{ij} + \gamma E V_{t+1}(\alpha\mathbf{u}^* + \mathbf{D} + (\beta-\alpha)\epsilon\mathbf{e}_i^n, \beta\mathbf{v}^* + \mathbf{S}) - \gamma E V_{t+1}(\alpha\mathbf{u}^* + \mathbf{D}, \beta\mathbf{v}^* + \mathbf{S}) \\
\geq& r_{ij}\epsilon + (h+c)\epsilon - \gamma\beta\epsilon r_{ij} - \gamma c\epsilon \geq 0,
\end{aligned}$$

which demonstrates that $\bar{\mathbf{Q}}$ weakly dominates $\mathbf{Q}^*$. Similarly, we can reach the same conclusion if $\alpha > \beta$.

Case (ii): Suppose that $u_i^* > 0$ and $v_j^* = 0$. By Theorem 3.1, under the conditions that $(i,j) \succeq (i',j)$ for all $i' \in \mathcal{D}$, we know that $q_{i'j}^* = 0$ for any $i' \in \mathcal{D}$ and $i' \neq i$. Then, $0 = v_j^* = y_j - \sum_{i'=1}^n q_{i'j}^* = y_j - q_{ij}^*$. Thus, $q_{ij}^* = y_j \geq \min\{x_i, y_j\}$, implying that $q_{ij}^* = \min\{x_i, y_j\}$ because $q_{ij}^* \leq \min\{x_i, y_j\}$.

Similarly, we can prove $q_{ij}^* = \min\{x_i, y_j\}$, if $u_i^* = 0$ and $v_j^* > 0$.

Case (iii): $u_i^* = v_j^* = 0$. Assume $q_{ij}^* < \min\{x_i, y_j\}$. Then, there must exist $j' \neq j$ and $i' \neq i$ such that $q_{ij'}^* > 0$ and $q_{i'j}^* > 0$. We choose $\epsilon > 0$ such that $q_{i'j}^* - \epsilon > 0$ and $q_{ij'}^* - \epsilon > 0$ and define $\bar{\mathbf{Q}}$ as $\tilde{\mathbf{Q}} = \mathbf{Q}^* + \epsilon(\mathbf{e}_{ij}^{n\times m} + \mathbf{e}_{i'j'}^{n\times m} - \mathbf{e}_{i'j}^{n\times m} - \mathbf{e}_{ij'}^{n\times m})$. The decision $\bar{\mathbf{Q}}$ is feasible, because $\bar{\mathbf{Q}} \geq \mathbf{0}$ and the post-matching levels of $\bar{\mathbf{Q}}$ are the same as that of $\mathbf{Q}^*$. Then, $H_t(\bar{\mathbf{Q}}, \mathbf{x}, \mathbf{y}) - H_t(\mathbf{Q}^*, \mathbf{x}, \mathbf{y}) = \epsilon(r_{ij} + r_{i'j'} - r_{i'j} - r_{ij'}) \geq 0$, implying that $\bar{\mathbf{Q}}$, in which $\bar{q}_{ij} = q_{ij}^* + \epsilon$, weakly dominates $\mathbf{Q}^*$. Following the same argument, we can always find an optimal decision $\bar{\mathbf{Q}}$ in which $\bar{q}_{ij} = \min\{x_i, y_j\}$. □

Proof of Theorem 3.3 For any matrix $(a_{ij})_{i\in\mathcal{D}, j\in\mathcal{S}}$, we rewrite $a_{i\cdot} = \sum_{j\in\mathcal{S}} a_{ij}$ and $a_{\cdot j} = \sum_{i\in\mathcal{D}} a_{ij}$. We first the following lemma.

Lemma 3.4 *For a matching decision $\mathbf{Q} = (q_{ij})_{i\in\mathcal{D}, j\in\mathcal{S}}$ that may not be feasible for the state $(\mathbf{x}', \mathbf{y}')$ in period t, we can find a matching decision $\mathbf{Q}'' = (q_{ij} - \delta_{ij} - \delta'_{ij})_{i\in\mathcal{D}, j\in\mathcal{S}}$ feasible for $(\mathbf{x}', \mathbf{y}')$, such that $\delta_{ij} \geq 0$ and $\delta'_{ij} \geq 0$ for all $i \in \mathcal{D}$ and $j' \in \mathcal{S}$, $\sum_{j'\in\mathcal{S}} \delta_{ij'} = (q_{i\cdot} - x'_i)^+$, and $\sum_{i'\in\mathcal{D}} \delta'_{i'j} = (q_{\cdot j} - \delta_{\cdot j} - y'_j)^+$.*

Proof of Lemma 3.4 For an $i \in \mathcal{D}$ and an arbitary permutation $(j_1, \ldots, j_m)$ of $\mathcal{S}$, we let $\delta_{ij_1} = \min\{(q_{i\cdot} - x'_i)^+, q_{ij_1}\}$, $\delta_{ij_2} = \min\{(q_{i\cdot} - x'_i)^+ - \delta_{ij_1}, q_{ij_2}\}$, …, $\delta_{ij_k} = \min\{(q_{i\cdot} - x'_i)^+ - \sum_{\ell=1}^{k-1} \delta_{ij_\ell}, q_{ij_k}\}$, …, $\delta_{ij_m} = \min\{(q_{i\cdot} - x'_i)^+ - \sum_{\ell=1}^{m-1} \delta_{ij_\ell}, q_{ij_m}\}$. It is easy to show that $(q_{i\cdot} - x'_i)^+ - \sum_{\ell=1}^{m-1} \delta_{ij_\ell} = (q_{ij_m} - x'_i)^+ \leq q_{ij_m}$. Thus $\delta_{ij_m} = (q_{i\cdot} - x'_i)^+ - \sum_{\ell=1}^{m-1} \delta_{ij_\ell}$, which implies that $\sum_{j'\in\mathcal{S}} \delta_{ij'} = \sum_{\ell=1}^{m} \delta_{ij_\ell} = (q_{i\cdot} - x'_i)^+$.

Let $q'_{ij} = q_{ij} - \delta_{ij}$ for all $j \in \mathcal{S}$. Then, $\sum_{j'\in\mathcal{S}} q'_{ij'} = \sum_{j'\in\mathcal{S}} q_{ij'} - \sum_{j'\in\mathcal{S}} \delta_{ij'} = q_{i\cdot} - \sum_{j'\in\mathcal{S}} \delta_{ij'} = q_{i\cdot} - (q_{i\cdot} - x'_i)^+ = \min\{q_{i\cdot}, x'_i\} \leq x'_i$.

Next, for $j \in \mathcal{S}$ and a permutation $(i_1, \ldots, i_n)$ of $\mathcal{D}$, we let $\delta'_{i_1 j} = \min\{(q'_{\cdot j} - y'_j)^+, q'_{i_1 j}\}$, …, $\delta'_{i_k j} = \min\left\{(q'_{\cdot j} - y'_j)^+ - \sum_{\ell=1}^{k-1} \delta'_{i_\ell j}, q_{q_{i_k j'}}\right\}$, …, $\delta'_{i_n j} = \min\{(q'_{\cdot j} - y'_j)^+ - \sum_{\ell=1}^{n-1} \delta'_{i_\ell j}, q_{i_n j'}\}$, and $q''_{i'j} = q'_{i'j} - \delta'_{i'j}$ for all $i' \in \mathcal{D}$. Again, we can show that $\sum_{\ell=1}^{n} \delta'_{i_\ell j} = (q'_{\cdot j} - y'_j)^+$ and that $\sum_{i'\in\mathcal{D}} q''_{i'j} = q'_{\cdot j} - \sum_{\ell=1}^{n} \delta'_{i_\ell j} =$

$\min\{q'_{\cdot j}, y'_j\} \le y'_j$. Therefore, $\mathbf{Q}'' = (q''_{ij})_{i\in\mathcal{D},\mathcal{S}}$ is a feasible decision with the desired properties. □

From a solution $\{\hat{q}_{ijt}\}_{i\in\mathcal{D},j\in\mathcal{S},t=\tau,\dots,T}$ to the problem $(\mathrm{P}_\tau^{\mathbf{x},\mathbf{y}})$ under the initial state $(\mathbf{x},\mathbf{y})$, we construct a feasible policy $\{\tilde{q}_{ijt}\}_{i\in\mathcal{D},j\in\mathcal{S},t=\tau,\dots,T}$, which we call policy M, from period τ to T as follows.

Step 1 Let $t \leftarrow \tau$. Define $\tilde{x}_{i\tau} = x_{i\tau} = x_i$, $\tilde{y}_{j\tau} = y_{j\tau} = y_j$ and $\tilde{q}_{ij\tau} = \hat{q}_{ij\tau}$ for all $i \in \mathcal{D}, j \in \mathcal{S}$;

Step 2 Let $t \leftarrow t+1$. Moreover,

- $\tilde{x}_{it} = \alpha(\tilde{x}_{i,t-1} - \sum_{j'\in\mathcal{S}} \tilde{q}_{ij'}) + D_{it}$, $\tilde{y}_{jt} = \beta(\tilde{y}_{j,t-1} - \sum_{i'\in\mathcal{D}} \tilde{q}_{i'j}) + S_{jt}$;
- Construct $\{\delta_{ijt}, \delta'_{ijt}\}_{i\in\mathcal{D},j\in\mathcal{S}}$ as in Lemma 3.4, a feasible decision $\tilde{\mathbf{Q}}_t = \left(\hat{q}_{ijt} - \delta_{ijt} - \delta'_{ijt}\right)_{i\in\mathcal{D},j\in\mathcal{S}}$ for the state $(\tilde{\mathbf{x}}_t, \tilde{\mathbf{y}}_t)$ in period t;
- Go to Step 2 if $t < T$; stop if $t = T$.

The following lemma bounds the gap between $V_\tau^{det}(\mathbf{x},\mathbf{y})$, the optimal value of the deterministic model, and $\tilde{V}_\tau(\mathbf{x},\mathbf{y})$, the value under policy M in the stochastic system.

Lemma 3.5 *We have*

$$\begin{aligned}
0 \le V_\tau^{det}(\mathbf{x},\mathbf{y}) - \tilde{V}_\tau(\mathbf{x},\mathbf{y}) \le & \sum_{i\in\mathcal{D}} (\max_{j\in\mathcal{S}} r_{ij}) \sum_{t'=\tau+1}^{t} E(\lambda_{it'} - D_{it'})^+ + \sum_{j\in\mathcal{S}} (\max_{i\in\mathcal{D}} r_{ij}) \sum_{t'=\tau+1}^{t} E(\mu_{jt'} - S_{jt'})^+ \\
& + c\sum_{t=\tau}^{T} \gamma^{t-\tau} \sum_{j\in\mathcal{S}} \sum_{t'=\tau+1}^{t} E(\mu_{jt'} - S_{jt'})^+ + c\sum_{t=\tau}^{T} \gamma^{t-\tau} \sum_{i\in\mathcal{D}} \sum_{t'=\tau+1}^{t} \alpha^{t-t'} E(D_{it'} - \lambda_{it'})^+ \\
& + h\sum_{t=\tau}^{T} \gamma^{t-\tau} \sum_{i'\in\mathcal{D}} \sum_{t'=\tau+1}^{t} E(\lambda_{it'} - D_{it'})^+ + h\sum_{t=\tau}^{T} \gamma^{t-\tau} \sum_{j\in\mathcal{S}} \sum_{t'=\tau+1}^{t} \beta^{t-t'} E(S_{jt'} - \mu_{jt'})^+.
\end{aligned}$$

Proof of Lemma 3.5 We have

$$\hat{u}_{it} - \tilde{u}_{it} = (\hat{x}_{it} - \hat{q}_{i\cdot,t}) - (\tilde{x}_{it} - \tilde{q}_{i\cdot,t}) = (\hat{x}_{it} - \tilde{x}_{it}) - \hat{q}_{i\cdot,t} + \tilde{q}_{i\cdot,t} = (\hat{x}_{it} - \tilde{x}_{it}) - \sum_{j'\in\mathcal{S}} (\delta_{ij't} + \delta'_{ij't}),$$

and

$$\begin{aligned}
\hat{x}_{i,t+1} - \tilde{x}_{i,t+1} = \alpha(\hat{u}_{it} - \tilde{u}_{it}) + \lambda_{i,t+1} - D_{i,t+1} = & \alpha(\hat{x}_{it} - \tilde{x}_{it}) - \alpha\sum_{j'\in\mathcal{S}} \delta_{ij't} - \alpha\sum_{j'\in\mathcal{S}} \delta'_{ij't} + \lambda_{i,t+1} - D_{i,t+1} \\
\le & \alpha[(\hat{x}_{it} - \tilde{x}_{it})^+ - \sum_{j'\in\mathcal{S}} \delta_{ij't}] + \lambda_{i,t+1} - D_{i,t+1} \\
\le & (\hat{x}_{it} - \tilde{x}_{it})^+ - \sum_{j'\in\mathcal{S}} \delta_{ij't} + \lambda_{i,t+1} - D_{i,t+1}, \qquad (3.13)
\end{aligned}$$

where the last inequality holds because $\sum_{j'\in\mathcal{S}} \delta_{ij't} = (\hat{q}_{i\cdot t} - \tilde{x}_{it})^+ \le (\hat{x}_{it} - \tilde{x}_{it})^+$ by Lemma 3.4.

We now show by induction that $(\hat{x}_{it} - \tilde{x}_{it})^+ \le -\sum_{t'=\tau}^{t-1} \sum_{j'\in\mathcal{S}} \delta_{ij't'} + \sum_{t'=\tau+1}^{t} (\lambda_{it'} - D_{it'})^+$. The inequality holds for $t = \tau$ as $\hat{x}_{i\tau} = \tilde{x}_{i\tau} = x_{i\tau}$. Suppose it also holds for a certain t. By (3.13) we have

$$\begin{aligned}(\hat{x}_{i,t+1} - \tilde{x}_{i,t+1})^+ \leq& (\hat{x}_{it} - \tilde{x}_{it})^+ - \sum_{j' \in \mathcal{S}} \delta_{ij't} + (\lambda_{i,t+1} - D_{i,t+1})^+ \\ \leq& - \sum_{t'=\tau}^{t-1} \sum_{j' \in \mathcal{S}} \delta_{ij't'} + \sum_{t'=\tau+1}^{t} (\lambda_{it'} - D_{it'})^+ - \sum_{j' \in \mathcal{S}} \delta_{ij't} + (\lambda_{i,t+1} - D_{i,t+1})^+ \\ =& - \sum_{t'=\tau}^{t} \sum_{j' \in \mathcal{S}} \delta_{ij't'} + \sum_{t'=\tau+1}^{t+1} (\lambda_{it'} - D_{it'})^+,\end{aligned}$$

which completes the induction.

It follows that $\sum_{j' \in \mathcal{S}} \delta_{ij't} = (\hat{q}_{i\cdot} - \tilde{x}_{it}) \leq (\hat{x}_{it} - \tilde{x}_{it})^+ \leq - \sum_{t'=\tau}^{t-1} \sum_{j' \in \mathcal{S}} \delta_{ij't'} + \sum_{t'=\tau+1}^{t} (\lambda_{it'} - D_{it'})^+$, or equivalently,

$$\sum_{t'=\tau}^{t} \sum_{j' \in \mathcal{S}} \delta_{ij't'} \leq \sum_{t'=\tau+1}^{t} (\lambda_{it'} - D_{it'})^+. \tag{3.14}$$

Analogously, we can show that

$$\sum_{t'=\tau}^{t} \sum_{i' \in \mathcal{D}} \delta'_{i'jt'} \leq \sum_{t'=\tau+1}^{t} (\mu_{jt'} - S_{jt'})^+. \tag{3.15}$$

Moreover,

$$\begin{aligned}\tilde{x}_{i,t+1} - \hat{x}_{i,t+1} =& \alpha(\tilde{x}_{it} - \hat{x}_{it}) + \alpha \sum_{j' \in \mathcal{S}} \delta_{ij't} + \alpha \sum_{j' \in \mathcal{S}} \delta'_{ij't} + D_{i,t+1} - \lambda_{i,t+1} \\ =& \alpha(\tilde{x}_{it} - \hat{x}_{it}) + \alpha(\hat{q}_{i\cdot} - \tilde{x}_{it})^+ + \alpha \sum_{j' \in \mathcal{S}} \delta'_{ij't} + D_{i,t+1} - \lambda_{i,t+1} \\ \leq& \alpha(\tilde{x}_{it} - \hat{x}_{it}) + \alpha(\hat{x}_{it} - \tilde{x}_{it})^+ + \alpha \sum_{j' \in \mathcal{S}} \delta'_{ij't} + D_{i,t+1} - \lambda_{i,t+1} \\ =& \alpha(\tilde{x}_{it} - \hat{x}_{it})^+ + \alpha \sum_{j' \in \mathcal{S}} \delta'_{ij't} + D_{i,t+1} - \lambda_{i,t+1}.\end{aligned}$$

Then $(\tilde{x}_{i,t+1} - \hat{x}_{i,t+1})^+ \leq \alpha(\tilde{x}_{it} - \hat{x}_{it})^+ + \alpha \sum_{j' \in \mathcal{S}} \delta'_{ij't} + (D_{i,t+1} - \lambda_{i,t+1})^+$. By induction we have

$$(\tilde{x}_{it} - \hat{x}_{it})^+ \leq \sum_{t'=\tau}^{t-1} \alpha^{t-t'} \sum_{j' \in \mathcal{S}} \delta'_{ij't'} + \sum_{t'=\tau+1}^{t} \alpha^{t-t'} (D_{it'} - \lambda_{it'})^+.$$

Then,

$$\begin{aligned}\tilde{u}_{it} - \hat{u}_{it} = (\tilde{x}_{it} - \hat{x}_{it}) + \sum_{j' \in \mathcal{S}} \delta_{i'jt} + \sum_{j' \in \mathcal{S}} \delta'_{ij't} \leq& (\tilde{x}_{it} - \hat{x}_{it}) + (\hat{x}_{it} - \tilde{x}_{it})^+ + \sum_{j' \in \mathcal{S}} \delta'_{ij't} \\ =& (\tilde{x}_{it} - \hat{x}_{it})^+ + \sum_{j' \in \mathcal{S}} \delta'_{ij't}\end{aligned}$$

$$\leq \sum_{t'=\tau}^{t} \alpha^{t-t'} \sum_{j' \in \mathcal{S}} \delta'_{ij't'} + \sum_{t'=\tau+1}^{t} \alpha^{t-t'} (D_{it'} - \lambda_{it'})^{+}. \quad (3.16)$$

Likewise we can prove

$$\tilde{v}_{jt} - \hat{v}_{jt} \leq \sum_{t'=\tau}^{t} \beta^{t-t'} \sum_{i' \in \mathcal{D}} \delta_{i'jt'} + \sum_{t'=\tau+1}^{t} \beta^{t-t'} (S_{jt'} - \mu_{jt'})^{+}. \quad (3.17)$$

Finally,

$$\begin{aligned}
& V_{\tau}^{\det}(\mathbf{x}, \mathbf{y}) - \tilde{V}_{\tau}(\mathbf{x}, \mathbf{y}) \\
=& \sum_{t=\tau}^{T} \gamma^{t-\tau} \sum_{i \in \mathcal{D}, j \in \mathcal{S}} r_{ij} \hat{q}_{ijt} - \sum_{t=\tau}^{T} \gamma^{t-\tau} (c \sum_{i \in \mathcal{D}} \hat{u}_{it} + h \sum_{j \in \mathcal{S}} \hat{v}_{it}) \\
& - E \sum_{t=\tau}^{T} \gamma^{t-\tau} \sum_{i \in \mathcal{D}, j \in \mathcal{S}} r_{ij} \tilde{q}_{ijt} + E \sum_{t=\tau}^{T} \gamma^{t-\tau} (c \sum_{i \in \mathcal{D}} \tilde{u}_{it} + h \sum_{j \in \mathcal{S}} \tilde{v}_{jt}) \\
=& E \sum_{t=\tau}^{T} \gamma^{t-\tau} \sum_{i \in \mathcal{D}, j \in \mathcal{S}} r_{ij} (\hat{q}_{ijt} - \tilde{q}_{ijt}) + cE \sum_{t=\tau}^{T} \gamma^{t-\tau} \sum_{i \in \mathcal{D}} (\tilde{u}_{it} - \hat{u}_{it}) + hE \sum_{t=\tau}^{T} \gamma^{t-\tau} \sum_{j \in \mathcal{S}} (\tilde{v}_{jt} - \hat{v}_{jt}) \\
=& E \sum_{t=\tau}^{T} \gamma^{t-\tau} \sum_{i \in \mathcal{D}, j \in \mathcal{S}} r_{ij} (\delta_{ijt} + \delta'_{ijt}) + cE \sum_{t=\tau}^{T} \gamma^{t-\tau} \sum_{i \in \mathcal{D}} (\tilde{u}_{it} - \hat{u}_{it}) + hE \sum_{t=\tau}^{T} \gamma^{t-\tau} \sum_{j \in \mathcal{S}} (\tilde{v}_{jt} - \hat{v}_{jt}) \\
\leq& E \left[\sum_{i \in \mathcal{D}} (\max_{j \in \mathcal{S}} r_{ij}) \sum_{t=\tau}^{T} \sum_{j \in \mathcal{S}} \delta_{ijt} + \sum_{j \in \mathcal{S}} (\max_{i \in \mathcal{D}} r_{ij}) \sum_{t=\tau}^{T} \sum_{i \in \mathcal{D}} \delta'_{ijt} \right] \\
& + cE \sum_{t=\tau}^{T} \gamma^{t-\tau} \sum_{i \in \mathcal{D}} \sum_{t'=\tau}^{t} \alpha^{t-t'} \sum_{j' \in \mathcal{S}} \delta'_{ij't'} + cE \sum_{t=\tau}^{T} \gamma^{t-\tau} \sum_{i \in \mathcal{D}} \sum_{t'=\tau+1}^{t} \alpha^{t-t'} (D_{it'} - \lambda_{it'})^{+} \\
& + hE \sum_{t=\tau}^{T} \gamma^{t-\tau} \sum_{j \in \mathcal{S}} \sum_{t'=\tau}^{t} \beta^{t-t'} \sum_{i' \in \mathcal{D}} \delta_{i'jt'} + hE \sum_{t=\tau}^{T} \gamma^{t-\tau} \sum_{j \in \mathcal{S}} \sum_{t'=\tau+1}^{t} \beta^{t-t'} (S_{jt'} - \mu_{jt'})^{+} \\
\leq& \sum_{i \in \mathcal{D}} (\max_{j \in \mathcal{S}} r_{ij}) \sum_{t'=\tau+1}^{t} E(\lambda_{it'} - D_{it'})^{+} + \sum_{j \in \mathcal{S}} (\max_{i \in \mathcal{D}} r_{ij}) \sum_{t'=\tau+1}^{t} E(\mu_{jt'} - S_{jt'})^{+} \\
& + c \sum_{t=\tau}^{T} \gamma^{t-\tau} \sum_{j \in \mathcal{S}} \sum_{t'=\tau+1}^{t} E(\mu_{jt'} - S_{jt'})^{+} + c \sum_{t=\tau}^{T} \gamma^{t-\tau} \sum_{i \in \mathcal{D}} \sum_{t'=\tau+1}^{t} \alpha^{t-t'} E(D_{it'} - \lambda_{it'})^{+} \\
& + h \sum_{t=\tau}^{T} \gamma^{t-\tau} \sum_{i' \in \mathcal{D}} \sum_{t'=\tau+1}^{t} E(\lambda_{it'} - D_{it'})^{+} + h \sum_{t=\tau}^{T} \gamma^{t-\tau} \sum_{j \in \mathcal{S}} \sum_{t'=\tau+1}^{t} \beta^{t-t'} E(S_{jt'} - \mu_{jt'})^{+},
\end{aligned}$$

where the first inequality follows from (3.16) and (3.17), and the second one from (3.14) and (3.15). □

Let $\tilde{V}_{\tau}^{k}(\mathbf{x}, \mathbf{y})$ be the value function under policy M in system k. The next lemma investigates the performance of policy M as and the resolving heuritic in stochastic system k.

Lemma 3.6 *We have* $[V_{\tau}^{\det(k)}(\mathbf{x}, \mathbf{y}) - \tilde{V}_{\tau}^{k}(\mathbf{x}, \mathbf{y})]/k = O(1/\sqrt{k})$ *and* $[V_{\tau}^{k}(\mathbf{x}, \mathbf{y}) - V_{\tau}^{\text{resolve}(k)}(\mathbf{x}, \mathbf{y})]/k = O(1/\sqrt{k})$ *uniformly for all* $(\mathbf{x}, \mathbf{y})$.

Proof of Lemma 3.6 To prove the first inequality, we obtain from Lemma 3.5 that

$$
\begin{aligned}
&V_\tau^{\det(k)}(\mathbf{x},\mathbf{y}) - \tilde{V}_\tau^k(\mathbf{x},\mathbf{y}) \\
\le &\sum_{i\in\mathcal{D}}(\max_{j\in\mathcal{S}} r_{ij}) \sum_{t'=\tau+1}^{t} E[k\lambda_{it'} - D_{it'}(k)]^+ + \sum_{j\in\mathcal{S}}(\max_{i\in\mathcal{D}} r_{ij}) \sum_{t'=\tau+1}^{t} E[k\mu_{jt'} - S_{jt'}(k)]^+ \\
&+ c\sum_{t=\tau}^{T}\gamma^{t-\tau}\sum_{j\in\mathcal{S}}\sum_{t'=\tau+1}^{t} E[k\mu_{jt'} - S_{jt'}(k)]^+ + c\sum_{t=\tau}^{T}\gamma^{t-\tau}\sum_{i\in\mathcal{D}}\sum_{t'=\tau+1}^{t}\alpha^{t-t'}E[D_{it'}(k) - k\lambda_{it'}]^+ \\
&+ h\sum_{t=\tau}^{T}\gamma^{t-\tau}\sum_{i'\in\mathcal{D}}\sum_{t'=\tau+1}^{t} E[k\lambda_{it'} - D_{it'}(k)]^+ + h\sum_{t=\tau}^{T}\gamma^{t-\tau}\sum_{j\in\mathcal{S}}\sum_{t'=\tau+1}^{t}\beta^{t-t'}E[S_{jt'}(k) - \mu_{jt'}]^+.
\end{aligned}
$$

By the central limit theorem, $(D_{it'}(k) - k\lambda_{it'})/\sqrt{k} \xrightarrow{d} N(0,\sigma^2_{d,it'})$ and $(S_{jt'}(k) - k\mu_{it'})/\sqrt{k} \xrightarrow{d} N(0,\sigma^2_{s,jt'})$, where $\sigma_{d,it'}$ and $\sigma_{s,jt'}$ are the standard deviations of $D_{it'}$ and $S_{jt'}$, respectively. Hence

$$
\begin{aligned}
&\limsup_{k\to\infty}\frac{V_\tau^{\det(k)}(\mathbf{x},\mathbf{y}) - \tilde{V}_\tau^k(\mathbf{x},\mathbf{y})}{\sqrt{k}} = \limsup_{k\to\infty}\frac{V_\tau^{\det(k)}(\mathbf{x},\mathbf{y}) - \tilde{V}_\tau^k(\mathbf{x},\mathbf{y})}{\sqrt{k}} \\
\le &\sum_{i\in\mathcal{D}}(\max_{j\in\mathcal{S}} r_{ij}) \sum_{t'=\tau+1}^{t} EX^-_{it'} + \sum_{j\in\mathcal{S}}(\max_{i\in\mathcal{D}} r_{ij}) \sum_{t'=\tau+1}^{t} EY^-_{jt'} \\
&+ c\sum_{t=\tau}^{T}\gamma^{t-\tau}\sum_{j\in\mathcal{S}}\sum_{t'=\tau+1}^{t} EY^-_{jt'} + c\sum_{t=\tau}^{T}\gamma^{t-\tau}\sum_{i\in\mathcal{D}}\sum_{t'=\tau+1}^{t}\alpha^{t-t'}EX^+_{it'} \\
&+ h\sum_{t=\tau}^{T}\gamma^{t-\tau}\sum_{i'\in\mathcal{D}}\sum_{t'=\tau+1}^{t} EX^-_{it'} + h\sum_{t=\tau}^{T}\gamma^{t-\tau}\sum_{j\in\mathcal{S}}\sum_{t'=\tau+1}^{t}\alpha^{t-t'}EY^+_{jt'},
\end{aligned}
$$

where $X_{it'} \sim N(0,\sigma^2_{d,it'})$ and $Y_{jt'} \sim N(0,\sigma^2_{s,jt'})$. This implies that $[V_\tau^{\det(k)}(\mathbf{x},\mathbf{y}) - \tilde{V}_\tau^k(\mathbf{x},\mathbf{y})]/k = O(1/\sqrt{k})$, and the convergence is uniform for all $(\mathbf{x},\mathbf{y})$.

We prove the second inequality by induction. We have $[V_T^k(\mathbf{x},\mathbf{y}) - V_T^{\text{resolve}(k)}(\mathbf{x},\mathbf{y})]/k = 0$ as in period T we have a deterministic problem for even the stochastic system. Suppose that $[V_{t+1}^k(\mathbf{x},\mathbf{y}) - V_{t+1}^{\text{resolve}(k)}(\mathbf{x},\mathbf{y})]/k = O(1/\sqrt{k})$ uniformly for all $(\mathbf{x},\mathbf{y})$.

Consider a matching policy D1 that applies $\{\hat{q}_{ijt}\}_{i\in\mathcal{D},j\in\mathcal{S}}$ from solving $(\mathrm{P}_t^{\mathbf{x},\mathbf{y}})$ in period t but uses the optimal decisions from period $t+1$ on. Let $\bar{V}_t^k(\mathbf{x},\mathbf{y})$ be the total expected discounted reward minus cost in system k from period t to period T under this policy. Policy D1 coincides with policy M and the resolving heuristic in period as they both solve $(\mathrm{P}_t^{\mathbf{x},\mathbf{y}})$. From period $t+1$, policy D1 would dominate the other two by definition. Thus

$$
\begin{aligned}
0 \le &\bar{V}_t^k(\mathbf{x},\mathbf{y})/k - V_t^{\text{resolve}(k)}(\mathbf{x},\mathbf{y})/k \\
= &\gamma E V_{t+1}^k(\alpha\mathbf{u}+\mathbf{D}_t, \beta\mathbf{v}+\mathbf{S}_t)/k - \gamma E V_{t+1}^{\text{resolve}(k)}(\alpha\mathbf{u}+\mathbf{D}_t, \beta\mathbf{v}+\mathbf{S}_t)/k = O(\frac{1}{\sqrt{k}}),
\end{aligned} \tag{3.18}
$$

where $\mathbf{u}$ and $\mathbf{v}$ are the post-matching levels in period t, when the matching decision suggested by $(\mathrm{P}_\tau^{\mathbf{x},\mathbf{y}})$

is used in that period.

Because $V_t^k(\mathbf{x},\mathbf{y}) \le V_t^{\det(k)}(\mathbf{x},\mathbf{y})$ and $\bar{V}_\tau^k(\mathbf{x},\mathbf{y}) \ge \tilde{V}_\tau^k(\mathbf{x},\mathbf{y})$, we have

$$0 \le V_t^k(\mathbf{x},\mathbf{y})/k - \bar{V}_\tau^k(\mathbf{x},\mathbf{y})]/k \le V_t^{\det(k)}(\mathbf{x},\mathbf{y})/k - \tilde{V}_\tau^k(\mathbf{x},\mathbf{y})]/k = O(\frac{1}{\sqrt{k}}). \tag{3.19}$$

Combining (3.18) and (3.19) we have

$$0 \le \frac{V_t^k(\mathbf{x},\mathbf{y}) - V_t^{\text{resolve}(k)}(\mathbf{x},\mathbf{y})}{k} = V_t^k(\mathbf{x},\mathbf{y})/k - \bar{V}_\tau^k(\mathbf{x},\mathbf{y})]/k + \bar{V}_t^k(\mathbf{x},\mathbf{y})/k - V_t^{\text{resolve}(k)}(\mathbf{x},\mathbf{y})/k = O(\frac{1}{\sqrt{k}}),$$

which completes the induction. □

Finally, one can show that $V_\tau^k(\mathbf{x},\mathbf{y}) = kV_\tau(\mathbf{x}/k,\mathbf{y}/k)$, and thus $\lim_{k\to\infty} V_\tau^k(\mathbf{x},\mathbf{y})/k = \lim_{k\to\infty} V_\tau(\mathbf{x}/k,\mathbf{y}/k) = V_\tau(\mathbf{0},\mathbf{0})$. As a result, $[V_t^k(\mathbf{x},\mathbf{y}) - V^{\text{resolve}(k)}(\mathbf{x},\mathbf{y})]/V_t^k(\mathbf{x},\mathbf{y}) = \left\{[V_t^k(\mathbf{x},\mathbf{y}) - V^{\text{resolve}(k)}(\mathbf{x},\mathbf{y})]/k\right\} / \left[V_t^k(\mathbf{x},\mathbf{y})/k\right] = O(1/\sqrt{k})$. □

Chapter 4

Dynamic Type Matching: Horizontal and Vertical Types

4.1 Introduction

In this chapter, we revisit the model (3.1) in Chapter 3, with more specific forms of matching reward structure. In particular, we study the following two cases of the general reward structure: unidirectionally horizontal types and vertical types. For these two reward structures, all neighboring pairs of demand and supply types are shown to be comparable (i.e., one dominating the other) under the modified Monge partial order, and priority is determined in an intuitive way: in the horizontal case, "distance" determines priority for matching a supply/demand type with different demand/supply types, and in the vertical case, "quality" determines priority. We follow the notation introduced in in Chapter 3.

Unidirectionally horizontal types. We assume that demand and supply types are located on a line or a circle. The *unidirectional* "distance" between a demand type and a supply type is the distance one travels unidirectionally along the line or circle from the location of the supply type to that of the demand type. The reward for matching two locations decreases linearly in their "distance." Using the general priority properties, we verify that it is optimal to match as much as possible the two that are closest to each other. Moreover, there exists a priority hierarchy in matching imperfect pairs. For any given demand (or supply) type, the closer its distance to a supply (or demand) type, the higher the priority to match the closer pair.[1] As a result, the optimal matching policy has a *match-down-to* structure: along the priority matching hierarchy, for a pair of demand and supply types, there exist state-dependent thresholds, with those for perfect pairs all equal to zero; if demand and supply levels are higher than the thresholds, they should be matched down to the thresholds; otherwise, they should

[1] Unfortunately, these results on priority, determined by distances, in general fail to hold if the "distance" is the shortest distance. As a result, one should not optimistically expect a general priority structure to hold for those situations.

not be matched.

Vertical types. Each demand or supply type is associated with a quality, and generates a higher reward if matched with a supply or demand type of a higher quality. In particular, we assume the reward of matching a pair is the sum of the contributions brought in by its components, which are increasing in quality. Then the optimal matching policy follows a simple structure, which we call *top-down matching* (in an economic term, assortative mating): line up demand types and supply types in descending order of their "quality" from high to low; match them from the top, down to some level. Thus, the optimal matching policy in any period can be fully determined by a total matching quantity. Moreover, we can take a dynamic perspective on the optimal matching policy: as in the case of horizontal types, in the top-down matching procedure there are match-down-to levels (or equivalently, some protection levels) for any pair of demand and supply types. When demand and supply have the same carry-over rate, we show, by verifying the $L^{\natural}$-concavity of the value functions of a transformed problem, that the optimal total matching quantity (from the aggregate perspective) or the optimal protection levels (from the dynamic perspective) have monotonicity properties with respect to the system state: An increment in the level of a demand or supply type with higher "quality" leads to a higher optimal matching quantity or lower protection levels.

The two cases with unidirectionally horizontal types and with vertical types apply to many emerging settings and also include many existing problems as special cases.

In particular, the case of unidirectionally horizontal types has the following applications:

Capacity management with upgrading. Upgrading uses a high-class supply to fulfill a low-class demand, which is widely adopted in the business practice, e.g., in travel industries (see, e.g., Yu et al. 2015) and in production/inventory settings (see, e.g., Bassok et al. 1999). Shumsky and Zhang (2009) study a revenue management problem with fixed initial capacities of various supply types, and demand types can only be upgraded one level higher. Yu et al. (2015) study a revenue management problem with fixed initial capacities of various supply types, and demand types can be upgraded to be matched with a higher-quality supply type. The upgrading reward structure in Yu et al. (2015) is a special case of unidirectionally horizontal types located along a line. Thus our results are applicable to a generalized capacity management problem with general upgrading and random replenishment. The feature of random supply is desirable for upgrading, even for those revenue management settings, not to mention for the production/inventory settings. For example, in car rental, car returns can be random and in airline ticket selling, early cancellations or airplane swaps can result in random capacity changes.

Carpooling/load matching along a fixed route. Roadie, an online platform, aims to entice college students and other travelers to earn extra pocket money by delivering large, long-haul items on the way to where they are already going. Platforms such as uShip and Cargomatic feed loads to independent truck drivers along their way. Carpooling platforms such as UberPool match riders heading to the same destination (or in the same direction). In those cases, the matching reward has two additive components: The first one is a disutility associated with the distance traveled along the fixed route from the driver's current location to pick up the demand.[2] The second is a utility associated with traveling along the route from the demand's pick-up location to its drop-off location. The former is the unidirectionally horizontal case, whereas the latter is a vertically differentiated attribute, because given the same pick-up location, it is more desirable if the demand's travel distance is longer. We show that if riders and drivers head to the same destination at the end of the route[3], a shorter distance to pick up a rider on the way has a higher priority in matching (see Section 4.2.2).

Type mating. A common feature of many manufacturing processes is the mating of two halves to produce a final product. For a flat panel display, the two halves are "active" and "passive" layers of an electronic display. For a ball bearing, the two halves are an inner race and outer race. The location of defects on each half can be examined and the mating of the two that have defects at the same locations generates the highest value. Duenyas et al. (1997) is the first to study this dynamic type mating problem. They assume that *one* unit of a demand and a supply type arrives in each period, and solves for the optimal mating policy that minimizes the long-run average cost. Baccara et al. (2016) study a similar problem but compare the centralized model with the decentralized model. Our model with horizontal types generalizes Duenyas et al. (1997) and Baccara et al. (2016) by accounting for arbitrary patterns of demand and supply arrivals.

The case of vertical types has the following ramifications and applications:

Assortative mating. In an empirical study of the centralized medical residency assignment, Agarwal (2015) assumes a simplified "double-vertical" model in which both the residents and programs have a (preference) utility function that is linear in observable traits of types on the other side. As a result, the matching reward for a pair is in the additive form, as assumed in the case of vertical types. The additive reward form is a special case of supermodular reward functions. For a one-shot setting, a supermodular reward function guarantees assortative mating as a stable matching. That is, high-quality demands are matched with high-quality supplies and low-quality demands with low-quality supplies, and types not matched to each other could not be mated without making at least one of them worse off. Surprisingly, this self-centric assortative mating behavior is also optimal for the centralized planner (Becker 2009, p.114). We show that in a dynamic setting, with the additive reward function, it is optimal for the centralized planner to perform top-down matching, i.e., assortative mating, up to some level, and save the rest for future. This result provides the following insights for a dynamic matching market: First, if the reward function is a general supermodular function other than an additive one, there exist scenarios in which socially efficient matching is not assortative (which seems not revealed before; see Li 2008 for a survey). This phenomenon happens only if the waiting/holding costs are moderate. This is because, in one extreme when the waiting and holding costs are sufficiently high, the intermediary prefers greedy

[2] A little detour can be allowed but tends to be negligible.

[3] In a press release on the launching of UberPool, Uber revealed that "on any given day, the vast majority of uberX trips in NYC have a 'lookalike' trip—a trip that starts near, ends near, and is happening around the same time as another trip"; see https://newsroom.uber.com/us-new-york/vision-for-the-future-1m-fewer-cars-on-the-road/.

matching within the current period, and assortative mating would emerge, and in the other extreme when the waiting/holding costs are close to zero, the intermediary prefers to hold up the matching until the last period with a bigger pool, and again assortative mating would emerge. Second, it may not be efficient to exhaust all demand and supply types at a given time. As a result, a centralized dating agency, or even a decentralized dating website, may want to limit the number of matching pairs at any time, in anticipation of future arrivals of better-quality men and women.

Inventory management with substitution. Consider that a firm sells a line of vertically differentiated products to multiple demand classes. Customers with their class is known to the firm are flexible with substitution, but is only willing to pay more for a higher-class product or will be compensated for a lower-class product, based on their class. The resulting reward function is in an additive form. Our results on vertical types are readily applicable to this dynamic inventory management setting with substitution and random supply.

Inventory rationing. The literature of inventory rationing considers inter-temporal inventory allocation of a single supply type across multiple demand types (see, e.g., Evans 1968, Veinott 1965, Ha 1997a,b and de Véricourt et al. 2002). Demand from the less valuable types can be rejected given possible future arrivals of demand from the more valuable types. The case of vertical types generalizes the idea of inventory rationing by considering *multiple* supply types (with exogenously random replenishment) and multiple demand types with general abandonment rates, whereas the literature considers *one* supply type (though with endogenized ordering/production decisions) and fully backlogged or lost demand types.

4.2 Horizontally Differentiated Types

In this section, we first consider the model with demand and supply types that are *horizontally* differentiated in the sense that each type has its own heterogeneous "taste."

4.2.1 The Directed Line Segment, Directed Circle and Undirected Line Segment

We assume that the n demand and m supply types are distributed on a fixed route C (e.g. a line segment) with a given direction. All the demand types have distinct locations (otherwise we can simply treat two demand types sharing the same location as the same type) and so do the supply types. For any two types t_1, t_2, we write $t_1 \to t_2$ to denote that t_1 is located before t_2, along the given direction. We denote by $\vec{d}(t_1, t_2)$ the travel distance from the location of t_1 to that of t_2 along the given direction.

The unit matching reward r_{ij} between type i demand and type j supply is a nonincreasing function of the distance between the two types, which is measured as follows. For $i \in \mathcal{D}$ and $j \in \mathcal{S}$ such that $j \to i$, we define $d_{ij} = \vec{d}(j, i)$. For $i \in \mathcal{D}$ and $j \in \mathcal{S}$ such that $i \to j$, we consider one of the following definitions:

(i) (Directed line segment) $d_{ij} = N$, where N is an arbitrarily large number;

(ii) (Directed circle) $d_{ij} = |C| - \vec{d}(i,j)$, where $|C|$ is the length of the route C;

(iii) (Undirected line segment) $d_{ij} = \vec{d}(i,j)$.

In case (i), a supply type is not allowed to travel counter to the given direction. In this case, C is a directed line segment, on which $i \in \mathcal{D}$ and $j \in \mathcal{S}$ can be matched with each other if and only if $j \to i$. The product upgrading model has the structure of a directed line segment.[4] Figure 4.1 illustrates such a model that allows general upgrading (see Yu et al. 2015).

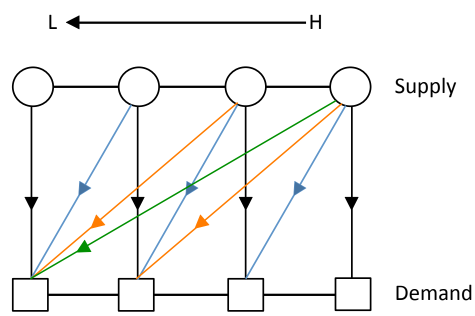


Figure 4.1: Product upgrade

In case (ii), a type j supply can travel along the given direction to reach a type i demand if $j \to i$. If $i \to j$, j needs to travel to the end of the route along the given direction, then "reappears" at the origin of the route and continues along the direction to reach i. This is equivalent to the case in which C is a directed (say, clockwise) circle and a supply type always needs to go clockwise on the circle to reach a demand type.

In case (iii), a supply type can go along or counter to the direction to reach a demand type. Thus the direction no longer plays a role and C is equivalent to an undirected line segment.

As mentioned, the unit matching reward can be written as $r_{ij} = f(d_{ij})$, where f is a nonincreasing function of the distance. If f takes a linear form, we can characterize the priorities in the optimal matching policy. The economic interpretation of a linear function f is that the reward r_{ij} from matching type i demand with type j supply is obtained from a base matching reward r_0 minus the mismatch cost proportional to the distance between type i demand and type j supply.

For $i \in \mathcal{D}$ and $j \in \mathcal{S}$, let $\overleftarrow{(i,j)}$ denote the segment of the route traveled by j to reach i. The following result shows the travel distance can imply the modified Monge partial order, hence implying the matching priority, in the light of Theorem 3.1.

Theorem 4.1 (Distance-based priority) *Suppose f is a linear and decreasing function.*

(i) If $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i,j')}$, then $(i,j) \succeq (i,j')$. Similarly, if $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j)}$, then $(i,j) \succeq (i',j)$.

[4]The model in Yu et al. (2015) corresponds to an incomplete bipartite network. While our base model considers a complete bipartite network structure, our results can be extended to the case where only a subset of the arcs are permissible.

(ii) In the case of undirected line segment, if $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j')}$ *and* $\overleftarrow{(i,j)}$ *has the same direction with* $\overleftarrow{(i',j')}$*, then* $(i,j) \succeq (i',j')$*.*

(iii) In the case of directed line segment and circle, $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j')}$ *is equivalent to* $(i,j) \succeq (i',j')$*.*

Proof of Theorem 4.1. (i) Without loss of generality, suppose that $r_{ij} = f(d_{ij}) = r_0 - d_{ij}$. The condition $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i,j')}$ implies that $d_{ij'} = d_{ij} + \vec{d}(j',j)$. Then, $r_{ij} - r_{ij'} = \vec{d}(j',j)$.

For any $i' \in \mathcal{D}$, $r_{i'j} - r_{i'j'} = d_{i'j'} - d_{i'j}$. To verify condition (D), it suffices to show that $r_{i'j} - r_{i'j'} \leq r_{ij} - r_{ij'}$, which is equivalent to $d_{i'j'} \leq d_{i'j} + \vec{d}(j',j)$ following the above arguments. The latter inequality is simply the triangular inequality. The other half can be proved analogously.

(ii) In the case of undirected line segment, if $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j')}$ and $\overleftarrow{(i,j)}$ has the same direction with $\overleftarrow{(i',j')}$, we have $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i,j')} \subseteq \overleftarrow{(i',j')}$. By part (i), $(i,j) \succeq (i,j') \succeq (i',j')$.

(iii) If $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j')}$, since $\overleftarrow{(i,j)}$ has the same direction with $\overleftarrow{(i',j')}$ is automatically satisfied for the directed line segment and directed circle, by part (ii), we have $(i,j) \succeq (i',j')$.

It remains to show that $(i,j) \succeq (i',j')$ implies $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j')}$. Suppose $(i,j) \succeq (i',j')$. We can assume without loss of generality that $(i,j) = (i_1,j_1) \succeq (i_1,j_2) \succeq (i_2,j_2) \succeq \cdots \succeq (i_\ell,j_\ell) = (i',j')$. By definition of the partial order, $(i_1,j_1) \succeq (i_1,j_2)$ implies $r_{i_1j_1} \geq r_{i_1j_2}$, thus $d_{i_1j_1} \leq d_{i_1j_2}$. In either the directed line segment case or the directed circle case, this suggests $\overleftarrow{(i_1,j_1)} \subseteq \overleftarrow{(i_1,j_2)}$. Repeating this argument we get $\overleftarrow{(i_1,j_1)} \subseteq \overleftarrow{(i_1,j_2)} \subseteq \overleftarrow{(i_2,j_2)} \subseteq \cdots \subseteq \overleftarrow{(i_l,j_l)}$. Thus $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j')}$. □

The next result shows that for the directed line segment and the directed circle, each demand or supply type should be matched in a greedy fashion with its closest match.

Theorem 4.2 (Greedy match of perfect pairs) *Consider the directed line segment case or the directed circle case. Suppose that* $\overleftarrow{(i,j)}$ *does not contain any other types than themselves. If* f *is nonincreasing and convex,* $q^*_{ij} = \min\{x_i, y_j\}$*.*

Proof of Theorem 4.2. If $\overleftarrow{(i,j)}$ does not contain any types other than i and j themselves, then $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j')}$ for any compatible pair (i',j') (in the case of directed line segment, we require $j' \to i'$; otherwise the arc (i',j') is "incompatible"). By Theorem 4.1 part (iii), $(i,j) \succeq (i',j')$. In particular, $(i,j) \succeq (i,j')$ and $(i,j) \succeq (i',j)$ for any $i' \in \mathcal{D}$ and $j' \in \mathcal{S}$. By Theorem 3.2, $i \in \mathcal{D}$ and $j \in \mathcal{S}$ should be matched with each other as much as possible. □

Given the analysis in this section so far, there exists an optimal hierarchy for the cases of the directed line segment and circle with linear reward functions. The optimal matching decision in a period can be characterized by state-dependent protection levels $a_{ij}(t,\cdot,\cdot)$ defined in a matching procedure as follows.

To start with, let $k = 1$, $(\mathbf{x}^1, \mathbf{y}^1) = (\mathbf{x}, \mathbf{y})$ and $\mathbf{Q}^* = \mathbf{0}^{n \times m}$. Also, we represent the set of arcs that have not been matched yet by $\bar{\mathcal{A}}$. Initially, $\bar{\mathcal{A}} = \mathcal{A}$.

Step 1. For each arc $(i,j) \in \left\{(i'',j'') \in \bar{\mathcal{A}} \mid \nexists (i',j') \in \bar{\mathcal{A}} \text{ such that } (i',j') \neq (i'',j'') \text{ and } \overleftarrow{(i',j')} \subseteq \overleftarrow{(i'',j'')}\right\}$ (i.e., (i,j) is undominated in $\bar{\mathcal{A}}$), we do the following.

Step 1.1. Match type i demand with type j supply until their remaining unmatched quantities reach $(x_i^k - y_j^k)^+ + a_{ij}(t, \mathbf{x}^k, \mathbf{y}^k)$ and $(x_i^k - y_j^k)^- + a_{ij}(t, \mathbf{x}^k, \mathbf{y}^k)$ respectively. Remove (i, j) from $\bar{\mathcal{A}}$. Set $q_{ij}^* = x_i^k - (x_i^k - y_j^k)^+ - a_{ij}(t, \mathbf{x}^k, \mathbf{y}^k)$.

Step 1.2. If $a_{ij}(t, \mathbf{x}^k, \mathbf{y}^k) > 0$, then set $q_{i'j'}^* = 0$ and remove (i', j') from $\bar{\mathcal{A}}$ for all $(i', j') \neq (i, j)$ such that $\overleftarrow{(i, j)} \subseteq \overleftarrow{(i', j')}$.

Step 2. Update the state vectors: $\mathbf{x}^{k+1} = \mathbf{x} - \mathbf{1}^m(\mathbf{Q}^*)^\intercal$, $\mathbf{y}^{k+1} = \mathbf{y} - \mathbf{1}^n\mathbf{Q}^*$. Increase k by 1. Go back to Step 1 if $\bar{\mathcal{A}}$ is nonempty, and stop otherwise.

The above procedure performs matching in a priority sequence, where k is the priority level. At a priority level k, the post-matching levels of type i demand and type j supply (right after the matching in Step 1) will be $(x_i^k - y_j^k)^+ + a_{ij}(t, \mathbf{x}^k, \mathbf{y}^k)$ and $(x_i^k - y_j^k)^- + a_{ij}(t, \mathbf{x}^k, \mathbf{y}^k)$ respectively. The level $a_{ij}(t, \mathbf{x}^k, \mathbf{y}^k)$ is the amount we would like to protect from matching. If $a_{ij}(t, \mathbf{x}^k, \mathbf{y}^k) > 0$, all arcs strictly dominated by (i, j) will have zero matching quantities due to the priority structure (see Step 1.2). When $k = 1$, each arc (i, j) chosen in Step 1 is undominated by any $(i', j') \in \mathcal{A}$, meaning that type i demand and type j supply will be matched as much as possible. Thus, $a_{ij}(t, \mathbf{x}^1, \mathbf{y}^1) = 0$ for all such (i, j). Another property of $a_{ij}(t, \mathbf{x}^k, \mathbf{y}^k)$ is that it depends on x_i^k and y_j^k only through their difference, $x_i^k - y_j^k$. This is because, if an arc (i, j) is ever selected in Step 1, the decision $\mathbf{Q}^*$ under the state $(\mathbf{x}, \mathbf{y})$ will lead to exactly the same post-matching levels as the decision $\mathbf{Q}^* + \epsilon\mathbf{e}_{ij}^{n\times m}$ under the state $(\mathbf{x} + \epsilon\mathbf{e}_i^n, \mathbf{y} + \epsilon\mathbf{e}_j^m)$. Since the total current-period rewards are linear in matching quantities, one can easily verify that $\mathbf{Q}^* + \epsilon\mathbf{e}_{ij}^{n\times m}$ will satisfy the first-order optimality conditions under the state $(\mathbf{x} + \epsilon\mathbf{e}_i^n, \mathbf{y} + \epsilon\mathbf{e}_j^m)$ if $\mathbf{Q}^*$ does so under the state $(\mathbf{x}, \mathbf{y})$. Consequently, $\mathbf{Q}^* + \epsilon\mathbf{e}_{ij}^{n\times m}$ is optimal for the state $(\mathbf{x} + \epsilon\mathbf{e}_i^n, \mathbf{y} + \epsilon\mathbf{e}_j^m)$ and has the same protection levels as $\mathbf{Q}^*$.

4.2.2 Additional Attribute

Consider an alternative reward function $r_{ij} = f(d_{ij}) + r_i^a$. The nonincreasing function $f(d_{ij})$ represents the reward associated with "traveling" by the type j supply to "reach" the type i demand, and r_i^a represents the additional reward related to the attribute of type i demand, e.g., the reward related to the traveling between pickup and drop-off locations of a customer. For this reward function, the same condition in part (i) of Theorem 4.1, $\overleftarrow{(i, j)} \subseteq \overleftarrow{(i, j')}$, is sufficient for $(i, j) \succeq (i, j')$. Then Theorem 3.1 would imply that in car sharing, for a given rider, a driver who is closer on the way should have a higher priority to be matched with that rider than another driver who is farther away. Moreover, in order for $(i, j) \succeq (i', j)$, it suffices to require $f(d_{ij}) + r_i^a \geq f(d_{i'j}) + r_{i'}^a$ in addition to $\overleftarrow{(i, j)} \subseteq \overleftarrow{(i', j)}$. To make implications of this result for carpooling, we caution a gap between the reality in which carpool drivers may drop off and pick up customers along the way and our implicit assumption that matched supply and demand would leave the system[5] (or the driver may come back but not in a way correlated with the current matching decisions). Under the assumption that a matched pair would leave the system, our result implies that for a given driver, if a rider is closer and has a longer travel distance, that rider should have a higher priority to be matched, as compared to another rider who is farther and has a shorter travel distance.

Commuting patterns of many cities indicate that drivers and riders often share the same destination. For example, the commuting pattern in the mornings of weekdays in New York City shows that commuters travel from different suburban areas in the same direction to the city.[6] In this case, the closer a rider to a driver, as they all go to the same destination, the higher the additional reward r_i^a earned after pickup. Hence, with the same-destination assumption, we can recover the same characterization of the optimal priority structure as Theorem 4.1(iii), for the alternative reward function $r_{ij} = f(d_{ij}) + r_i^a$.

Proposition 4.1 *Suppose that all supply and demand types are located on a directed line segment. Let e be the end point of that line segment, and that $r_i^a = g(\vec{d}(i,e))$ for all $i \in \mathcal{D}$, where g is a linear and nondecreasing function. With $r_{ij} = f(d_{ij}) + r_i^a$, where f is linear and nonincreasing, $(i,j) \succeq (i',j')$ is equivalent to $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j')}$.*

Proof of Proposition 4.1. First, we show that $(i,j) \succeq (i',j)$ if $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j)}$. If $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j)}$, then i' must be closer to the end point e, which leads to $r_i^a \geq r_{i'}^a$. Thus $r_{ij} = f(d_{ij}) + r_i^a \geq f(d_{i'j}) + r_{i'}^a = r_{i'j}$, given that $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j)}$. To verify Condition (D), we can use the fact that $r_{ij} + r_{i'j'} = f(d_{ij}) + f(d_{i'j'}) + r_i^a + r_{i'}^a \geq f(d_{ij'}) + f(d_{i'j}) + r_i^a + r_{i'}^a$, where the inequality holds because $f(d_{ij}) + f(d_{i'j'}) \geq f(d_{ij'}) + f(d_{i'j})$ that has been already proved in Theorem 4.1 part (i). Similarly, we can show that $(i',j) \succeq (i',j')$ if $\overleftarrow{(i',j)} \subseteq \overleftarrow{(i',j')}$. Then, for $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j')}$, we have $(i,j) \succeq (i',j) \succeq (i',j')$.

The same analysis as that of Theorem 4.1 part (iii) will show that $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j')}$ if $(i,j) \succeq (i',j')$. □

4.2.3 The Model with 2 Supply Types and 2 Demand Types

Next we sharpen the characterization of the optimal matching policy for the model with two demand and supply types, in which demand type i has the same location as supply type i, for $i = 1, 2$. Here, the types can be located on the directed line segment, directed circle, or undirected line segment. Let $\{i, -i\} = \{1, 2\}$ for $i = 1, 2$.

Observation 4.1 *For the 2-to-2 horizontal model, $(i,i) \succeq (i,-i)$ and $(i,i) \succeq (-i,i)$ for $i = 1, 2$.*

This is because we have $r_{ii} \geq \max\{r_{i,-i}, r_{-i,i}\}$ for $i = 1, 2$ (as long as f is nonincreasing function) even with d_{ij} defined as the shortest distance along the circle. As a result, by verifying the definition of the modified Monge partial order, it is easy to see that $(1,1) \succeq (1,2)$, $(1,1) \succeq (2,1)$, $(2,2) \succeq (2,1)$ and $(2,2) \succeq (1,2)$ hold. By Theorem 4.2, type i demand and type i supply, $i = 1, 2$, should be matched as much as possible. The matching in a period will stop if both supply types or both demand types are

[5] A driver who has multiple seats may still offer unmatched supply in the market.

[6] http://bigbytes.mobyus.com/commute.aspx

exhausted after being matched with their perfect match. Otherwise, it remains to decide the matching quantity between type i demand and type $-i$ supply if they have positive quantities left.

Theorem 4.3 (2-to-2 horizontal model: optimal matching policy) *Fix an arbitrary period t. For any $(\mathbf{x}, \mathbf{y})$, define the type-specific demand and supply imbalance $\eta_i \equiv x_i - y_i$ for $i = 1, 2$, and the aggregate imbalance $\eta \equiv \eta_1 + \eta_2$. The following matching procedure is optimal: for $i = 1, 2$,*

(i) Round 1 (Greedy matching for the perfect pair)*: match type i demand and supply as much as possible, i.e., $q^*_{ii} = \min\{x_i, y_i\}$.*

(ii) Round 2 ("Match down to" policy for the imperfect pair)*: if $x_i > y_i$ and $x_{-i} < y_{-i}$, then $q^*_{-i,i} = 0$ and match the imperfect pair of type i demand and type $-i$ supply down to post-matching levels $u^*_i = \eta^+ + a^*_i(t, \eta)$ and $v^*_{-i} = \eta^- + a^*_i(t, \eta)$ respectively, where $a^*_i(t, \eta) = \min\{\bar{a}_i(t, \eta), \eta_i - \eta^+\}$ and $\bar{a}_i(t, \eta)$ is some protection level. Otherwise, $q^*_{i,-i} = 0$.*

Proof of Theorem 4.3. By Observation 4.1 and Theorem 4.2, it is optimal to match the perfect pair as much as possible; that is done in the first round. Next we consider after round 1 how to match the imperfect pairs $(1, 2)$ and $(2, 1)$.

If $x_1 \geq y_1$ and $x_2 \geq y_2$ or $x_1 \leq y_1$ and $x_2 \leq y_2$, it is obvious that $q^*_{12} = q^*_{21} = 0$. Consider the case where $x_1 > y_1$ and $x_2 < y_2$. (The same argument applies to the case where $x_1 < y_1$ and $x_2 > y_2$.) After round 1, the remaining quantities for type 1 demand and type 2 supply is $x_1 - y_1 > 0$ and $y_2 - x_2 > 0$ respectively. There is no remaining unmatched type 2 demand and type 1 supply, and thus $q^*_{21} = 0$. It remains to determine the optimal matching quantity q^*_{12}, which is equivalent to determining some optimal protection level $a^*_1(t, \mathbf{x}, \mathbf{y})$. To see this, note that η^+ and η^- would be the unmatched type 1 demand and type 2 supply remaining after the imperfect pair $(1, 2)$ has been matched as much as possible, where $\eta = \eta_1 + \eta_2 = (x_1 - y_1) - (y_2 - x_2)$. When the protection level is a_1, the post-matching levels of type 1 demand and type 2 supply are $u_1 = \eta^+ + a_1$ and $v_2 = \eta^- + a_1$, respectively. The protection level needs to satisfy the nonnegativity constraint $a_1 \geq 0$ and ensure the matching quantity $q_{12} = \eta_1 - u_1 = \eta_1 - \eta^+ - a_1 \geq 0$, resulting in $a_1 \leq \eta_1 - \eta^+$. After Round 1, the cost-to-go function can be written in terms of the protection level a_1 as:

$$\begin{aligned}\widetilde{H}_t(a_1, \mathbf{x}, \mathbf{y}) =& r_{11}y_1 + r_{22}x_2 + r_{12}(\eta_1 - \eta^+ - a_1) - c(\eta^+ + a_1) - h(\eta^- + a_1)\\ &+ \gamma EV_{t+1}\left(\alpha(\eta^+ + a_1) + D_1, D_2, S_1, \beta(\eta^- + a_1) + S_2\right)\\ =& r_{11}y_1 + r_{22}x_2 + r_{12}(\eta_1 - \eta^+) - c\eta^+ - h\eta^- + \hat{H}_t(a_1, \eta),\end{aligned}$$

where

$$\hat{H}_t(a_1, \eta) = -(r_{12} + c + h)a_1 + \gamma EV_{t+1}\left(\alpha(\eta^+ + a_1) + D_1, D_2, S_1, \beta(\eta^- + a_1) + S_2\right) \tag{4.1}$$

depends on $(\mathbf{x}, \mathbf{y})$ only through η. The optimal protection level a^*_1 solves $\max_{0 \leq a_1 \leq \eta_1 - \eta^+} \hat{H}_t(a_1, \eta)$. As with the proof of Proposition 3.1, it is easy to show that $\hat{H}_t(a_1, \eta)$ is concave in a_1. Thus, the optimal protection level $a_1^*(t, \mathbf{x}, \mathbf{y}) = a_1^*(t, \eta) = \min\{\bar{a}_1(t, \eta), \eta_1 - \eta^+\}$, where $\bar{a}_1(t, \eta) \in \arg\max_{a_1 \geq 0} \hat{H}_t(a_1, \eta)$. □

Theorem 4.3 shows the structure of the optimal matching policy for the 2-to-2 horizontal model. In the first round of matching, type i demand is matched as much as possible with its most favorable match, type i supply. After that, if we matched the imperfect pair, type i demand and type $-i$ supply, to the full extent, then the post-matching levels of type i demand and type $-i$ supply would become η^+ and η^- respectively. The optimal matching quantity is characterized by the state-dependent *protection level* $\bar{a}_i(t, \eta)$: the amount $\bar{a}_i(t, \eta)$ is protected from being matched between type i demand and type $-i$ supply so that they are saved for the possible arrival of their perfect match in future periods. The *match-down-to* levels for type i demand and type $-i$ supply after the second round of matching are $\eta^+ + \bar{a}_i(t, \eta)$ and $\eta^- + \bar{a}_i(t, \eta)$ respectively. The matching of the imperfect pair has a match-down-to structure: If the quantity of type i demand, $\eta_i = x_i - y_i$, after the first round of matching is greater than the match-down-to level $\eta^+ + \bar{a}_i(t, \eta)$, then it is optimal to match the imperfect pair and bring the quantity of type i demand down to the level $\eta^+ + \bar{a}_i(t, \eta)$ (and simultaneously, that of type $-i$ supply down to $\eta^- + \bar{a}_i(t, \eta)$). Otherwise, type i demand and type $-i$ supply will not be matched. This structure is analogous to many threshold-type structures in the inventory literature, e.g., the celebrated base-stock policy. Moreover, the match-down-to levels only depend on the *aggregated* discrepancy between total demand and supply across two types. In other words, the match-down-to levels depend on the 4-dimensional state $(\mathbf{x}, \mathbf{y})$ *only* through a scalar η. We can obtain a further state collapse in the protection levels for the imperfect matching when the unmatched demand or supply is lost after the matching in each period is done.

Corollary 4.1 (2-to-2 horizontal model with lost demand or supply) *Suppose that $x_i > y_i$ and $x_{-i} < y_{-i}$. If $\alpha = 0$, there exists a constant $\hat{v}_{-i}(t)$ such that the optimal matching quantity between type i demand and type $-i$ supply is $q_{i,-i}^* = \eta_{-i}^- - \max\{\hat{v}_{-i}(t) \wedge \eta_{-i}^-, \eta^-\}$. If $\beta = 0$, there exists $\hat{u}_i(t)$ such that is $q_{i,-i}^* = \eta_i^+ - \max\{\hat{u}_i(t) \wedge \eta_i^+, \eta^+\}$.*

Proof of Corollary 4.1. Consider the case in which $x_1 \geq y_1$ and $x_2 \leq y_2$. If $\alpha = 0$, in the proof of Theorem 4.3, (4.1) reduces to $\hat{H}_t(a_1, \eta) = -(r_{12} + c + h)a_1 + \gamma EV_{t+1}(D_1, D_2, S_1, \beta(\eta^- + a_1) + S_2)$. To optimize the protection level a_1 is equivalent to optimizing the post-matching level $v_2 = \eta^- + a_1$ of supply type 2. Let $\hat{v}_2(t) \in \arg\max_{v_2 \geq 0} -(r_{12} + c + h)v_2 + \gamma EV_{t+1}(D_1, D_2, S_1, \beta v_2 + S_2)$, which is independent of η. Since $0 \leq a_1 \leq \eta_1 - \eta^+$, $\eta^- \leq v_2 = \eta^- + a_1 \leq \eta_2^-$. Then, $v_2^* = \max\{\hat{v}_2(t) \wedge \eta_2^-, \eta^-\}$, $a_1^*(t, \eta) = v_2^* - \eta^- = [\hat{v}_2(t) \wedge \eta_2^- - \eta^-]^+$ and $q_{12}^* = \eta_2^- - v_2^* = \eta_2^- - \max\{\hat{v}_2(t) \wedge \eta_2^-, \eta^-\}$. Analogously, we can show the desired result for $\beta = 0$. □

Corollary 4.1 says that if all the unmatched demand is lost, then the second-round matching reduces the quantity of type $-i$ supply *as close as possible* to a threshold $\hat{v}_{-i}(t)$, which is independent of η. Intuitively, because all the unmatched demand is lost and the post-matching level of supply type i has to be 0 after Round 1 of matching, the intermediary only cares about how much of supply type $-i$ to carry to the next period. This results in a *constant* protection level for supply type $-i$ for the current period. Similarly, if all the unmatched supply is lost, then the optimal matching policy reduces the quantity of type i demand as close as possible to the threshold $\hat{u}_i(t)$.

4.3 Vertically Differentiated Types

In this section, we consider *vertically* differentiated demand and supply types. Each demand or supply type is associated with a "quality" and generates a higher reward if it is matched with a supply or demand type of a higher quality. Specifically, we consider an additive form of the reward structure: for all $1 \le i \le n$ and $1 \le j \le m$, $r_{ij} = r_i^d + r_j^s$, where r_i^d (or r_j^s) can be understood as the quality of type i demand (or type j supply). Without loss of generality, we index the types such that $r_1^d > \cdots > r_n^d$ and $r_1^s > \cdots > r_m^s$. Agarwal (2015) assumes such an additive reward structure.

With the additive reward structure, $r_{ij} + r_{i'j'} = r_{ij'} + r_{i'j}$ for all $i, i' \in \mathcal{D}$ and $j, j' \in \mathcal{S}$. This implies that for two neighboring arcs, $(i,j) \succeq (i',j)$ if and only if $r_i^d \ge r_{i'}^d$, and $(i,j) \succeq (i,j')$ if and only if $r_j^s \ge r_{j'}^s$. This observation can easily be generalized as $(i,j) \succeq (i',j')$ if and only if $i < i'$ and $j < j'$. By Theorem 3.2, it is optimal to match type 1 demand and type 1 supply in a greedy fashion. From Theorem 3.1, the arc (i,j) has priority over (i,j') and (i',j) for all $j' > j$ and $i' > i$. This leads to an optimal policy that follows a top-down matching procedure (see Figure 4.2 for an illustration):

Corollary 4.2 (Top-down matching) *Line up demand types and supply types separately in increasing order of their indices. Match from the top, down to some level. The optimal matching decision* $\mathbf{Q}$ *in a period is fully determined by a total matching quantity* $Q \overset{def}{=} \sum_{i'=1}^{n} \sum_{j'=1}^{m} q_{i'j'}$.

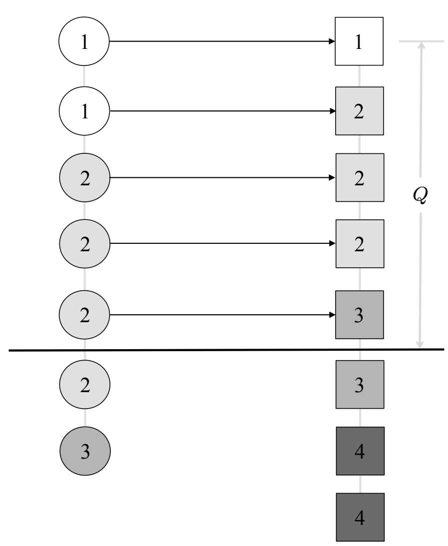


Figure 4.2: Line up, match up (to a "match-down-to" level).

Once Q is known, we can recover the matching matrix $\mathbf{Q}$ as follows: Starting with $i = 1$ and $j = 1$, we match type i demand with type j supply until one of them is fully matched or the total matching quantity reaches Q. If type i demand (or type j supply) is fully matched, we increase i (or j) by 1. Then we repeat the above steps until the total matching quantity finally reaches Q.

For ease of notation we define the following transformed state variables: $\tilde{x}_i \stackrel{\text{def}}{=} \sum_{i'=1}^{i} x_{i'}$ for $1 \le i \le n$, $\tilde{y}_j \stackrel{\text{def}}{=} \sum_{j'=1}^{j} y_{j'}$ for $1 \le i \le m$, and $\tilde{x}_0 \equiv \tilde{y}_0 \equiv 0$. If $\tilde{x}_{i-1} \le Q \le \tilde{x}_i$ and $\tilde{y}_{j-1} \le Q \le \tilde{y}_j$, then types $1, \ldots, i-1$ demand and types $1, \ldots, j-1$ supply are fully matched, an amount $Q - \tilde{x}_{i-1}$ of type i demand is matched with some supply, and an amount $Q - \tilde{y}_{j-1}$ of type j supply is matched with some demand. The rest of the types with quality lower than type i on the demand side and lower than type j on the supply side will not be matched in this period. The unmatched amount of type i' demand is $(\tilde{x}_{i'} - Q)^+ - (\tilde{x}_{i'-1} - Q)^+$, where the first term is the unmatched amount of demand in types $1, \ldots, i'$, and the second term is the unmatched amount of demand in types $1, \ldots, i'-1$. Thus a total amount of $x_{i'} - [(\tilde{x}_{i'} - Q)^+ - (\tilde{x}_{i'-1} - Q)^+]$ of type i' demand is matched with some supply in period t. Similarly, an amount $y_{j'} - [(\tilde{y}_{j'} - Q)^+ - (\tilde{y}_{j'-1} - Q)^+]$ of type j' supply is matched with some demand in period t. Thus, the total reward from matching in period t is

$$\begin{aligned}
&\sum_{i'=1}^{n} r_{i'}^d \left\{x_{i'} - [(\tilde{x}_{i'} - Q)^+ - (\tilde{x}_{i'-1} - Q)^+]\right\} + \sum_{j'=1}^{m} r_{j'}^s \left\{y_{j'} - [(\tilde{y}_{j'} - Q)^+ - (\tilde{y}_{j'-1} - Q)^+]\right\} \\
=&\sum_{i'=1}^{n} r_{i'}^d x_{i'} + \sum_{j'=1}^{m} r_{j'}^s y_{j'} - \sum_{i'=1}^{n} \left(r_{i'}^d - r_{i'+1}^d\right) (\tilde{x}_{i'} - Q)^+ - \sum_{j'=1}^{m} \left(r_{j'}^s - r_{j'+1}^s\right) (\tilde{y}_{j'} - Q)^+,
\end{aligned}$$

where $r_{n+1}^d = r_{m+1}^s \equiv 0$. The post-matching levels of demand and supply are given by $u_{i'} = v_{j'} = 0$ for $i' < i$ and $j' < j$, $u_i = \tilde{x}_i - Q$, $v_j = \tilde{y}_j - Q$, $\mathbf{u}_{[i+1,n]} = \mathbf{x}_{[i+1,n]} = (x_{i+1}, \ldots, x_n)$ and $\mathbf{v}_{[j+1,m]} = \mathbf{y}_{[j+1,m]} = (y_{j+1}, \ldots, y_m)$. Then we can rewrite the DP (3.1) as the following DP with a single decision variable Q:

$$\begin{aligned}
V_t(\mathbf{x}, \mathbf{y}) &= \max_{0 \le Q \le \min\{\tilde{x}_n, \tilde{y}_m\}} G_t(Q, \mathbf{x}, \mathbf{y}), \\
G_t(Q, \mathbf{x}, \mathbf{y}) &= \sum_{i'=1}^{n} r_{i'}^d x_{i'} + \sum_{j'=1}^{m} r_{j'}^s y_{j'} - \sum_{i'=1}^{n} \left(r_{i'}^d - r_{i'+1}^d\right) (\tilde{x}_{i'} - Q)^+ - \sum_{j'=1}^{m} \left(r_{j'}^s - r_{j'+1}^s\right) (\tilde{y}_{j'} - Q)^+ \\
&\quad - c(\tilde{x}_n - Q) - h(\tilde{y}_m - Q) + \gamma E V_{t+1}(\alpha \mathbf{u} + \mathbf{D}, \beta \mathbf{v} + \mathbf{S}),
\end{aligned} \tag{4.2}$$

where $r_{n+1}^d = r_{m+1}^s \equiv 0$, $\mathbf{u} = (\mathbf{0}^{i-1}, \tilde{x}_i - Q, \mathbf{x}_{[i+1,n]})$ and $\mathbf{v} = (\mathbf{0}^{j-1}, \tilde{y}_j - Q, \mathbf{y}_{[j+1,m]})$ if $\tilde{x}_{i-1} \le Q < \tilde{x}_i$ and $\tilde{y}_{j-1} \le Q < \tilde{y}_j$.

Lemma 4.1 *$G_t(Q, \mathbf{x}, \mathbf{y})$ is concave in Q.*

Proof of Lemma 4.1. It is easy to see that G_t is concave in Q within the interior of the ranges $\tilde{x}_{i-1} \le Q < \tilde{x}_i$ and $\tilde{y}_{j-1} \le Q < \tilde{y}_j$. Without loss of generality, we assume $\tilde{x}_i \in (\tilde{y}_{j-1}, \tilde{y}_j)$. We show that G_t is concave in the neighborhood of a breakpoint $a = \tilde{x}_i$. To this end, it suffices to show that $G_t(a + \epsilon, \mathbf{x}, \mathbf{y}) - G_t(a, \mathbf{x}, \mathbf{y}) \le G_t(a, \mathbf{x}, \mathbf{y}) - G_t(a - \epsilon, \mathbf{x}, \mathbf{y})$, where $0 < \epsilon < \min\{\tilde{x}_i - \tilde{y}_{j-1}, \tilde{y}_j - \tilde{x}_i\}$. Let $\mathbf{u} = (\mathbf{0}^i, \mathbf{x}_{[i+1,n]})$ and $\mathbf{v} = (\mathbf{0}^{j-1}, \tilde{y}_j - \tilde{x}_i, \mathbf{y}_{[j+1,m]})$. We have

$$G_t(a, \mathbf{x}, \mathbf{y}) - G_t(a - \epsilon, \mathbf{x}, \mathbf{y})$$

$$
\begin{aligned}
=&(r_i^d + r_j^s + c + h)\epsilon + \gamma EV_{t+1}(\alpha \mathbf{u} + \mathbf{D}, \beta \mathbf{v} + \mathbf{S}) - \gamma EV_{t+1}(\alpha \mathbf{u} + \alpha\varepsilon \mathbf{e}_i^n + \mathbf{D}, \beta \mathbf{v} + \beta\varepsilon \mathbf{e}_j^m + \mathbf{S}) \\
\geq&(r_i^d + r_j^s + c + h)\epsilon - \gamma\alpha(r_i^d - r_{i+1}^d)\epsilon \\
&+ \gamma EV_{t+1}(\alpha \mathbf{u} + \mathbf{D}, \beta \mathbf{v} + \mathbf{S}) - \gamma EV_{t+1}(\alpha \mathbf{u} + \alpha\varepsilon \mathbf{e}_{i+1}^n + \mathbf{D}, \beta \mathbf{v} + \beta\varepsilon \mathbf{e}_j^m + \mathbf{S}),
\end{aligned}
$$

where the inequality follows from Lemma 3.2 (set ϵ_t^2 to zero) and the fact that $-\sum_{j'=1}^m \lambda_{j'}(r_{ij'} - r_{i'j'}) = -\sum_{j'=1}^m \lambda_{j'}(r_i^d - r_{i'}^d) \geq -(r_i^d - r_{i'}^d)\alpha\epsilon$ if $\sum_{j'=1}^m \lambda_{j'} \leq \alpha\epsilon$. On the other hand, we have

$$
\begin{aligned}
&G_t(a+\epsilon, \mathbf{x}, \mathbf{y}) - G_t(a, \mathbf{x}, \mathbf{y}) \\
=&(r_{i+1}^d + r_j^s + c + h)\epsilon + \gamma EV_{t+1}(\alpha \mathbf{u} - \alpha\varepsilon \mathbf{e}_{i+1}^n + \mathbf{D}, \beta \mathbf{v} - \beta\varepsilon \mathbf{e}_j^m + \mathbf{S}) - \gamma EV_{t+1}(\alpha \mathbf{u} + \mathbf{D}, \beta \mathbf{v} + \mathbf{S}).
\end{aligned}
$$

Since $\gamma, \alpha \in [0,1]$ and $r_i^d \geq r_{i+1}^d$, $(r_i^d + r_j^s + c + h)\epsilon - \gamma\alpha(r_i^d - r_{i+1}^d)\epsilon \geq (r_i^d + r_j^s + c + h)\epsilon - (r_i^d - r_{i+1}^d)\epsilon = (r_{i+1}^d + r_j^s + c + h)\epsilon$. Then, by the concavity of $V_{t+1}(\cdot)$ (see Proposition 3.1), it follows that $G_t(a+\epsilon, \mathbf{x}, \mathbf{y}) - G_t(a, \mathbf{x}, \mathbf{y}) \leq G_t(a, \mathbf{x}, \mathbf{y}) - G_t(a-\epsilon, \mathbf{x}, \mathbf{y})$. □

By Lemma 4.1, the optimal matching decisions in a period become a one-dimensional convex optimization problem. The following result sharpens the optimal policy characterization. To facilitate the presentation, define $\overline{x}_i = x_i - (\tilde{y}_{j-1} - \tilde{x}_{i-1})^+$ as the available quantity of type i demand before we consider its matching with type j supply, and $\underline{x}_i = (\tilde{x}_i - \tilde{y}_j)^+$ as the remaining quantity of type i demand after we match it with type j supply as much as possible. We define $\overline{y}_j$ and $\underline{y}_j$ similarly.

Theorem 4.4 (Vertical model: optimal matching procedure) *Consider vertically differentiated types. In the top-down matching procedure, consider matching type i demand with type j supply, which is optimal only if all types $1, \ldots, i-1$ demand and types $1, \ldots, j-1$ supply have been fully matched, and $\tilde{x}_i > \tilde{y}_{j-1}$ and $\tilde{x}_{i-1} < \tilde{y}_j$. There exists a protection level $a_{ij}^*(t)$ depending on $(\tilde{x}_i - \tilde{y}_j, \mathbf{x}_{[i+1,n]}, \mathbf{y}_{[j+1,m]})$ such that it is optimal to match type i demand with type j supply until the level of type i demand reduces to $(\tilde{x}_i - \tilde{y}_j)^+ + a_{ij}^*(t)$ if $\overline{x}_i - \underline{x}_i > a_{ij}^*(t)$ (or equivalently, the level of type j supply reduces to $(\tilde{y}_j - \tilde{x}_i)^+ + a_{ij}^*(t)$ if $\overline{y}_j - \underline{y}_j > a_{ij}^*(t)$), and otherwise not to match type i demand with type j supply.*

Proof of Theorem 4.4. Take the dynamic view of the top-down matching procedure and consider the scenario when it gets to the matching of type i demand and type j supply. The available amount of type i demand is $\overline{x}_i \stackrel{\text{def}}{=} x_i - (\tilde{y}_{j-1} - \tilde{x}_{i-1})^+$ and that of type j supply is $\overline{y}_j \stackrel{\text{def}}{=} y_j - (\tilde{x}_{i-1} - \tilde{y}_{j-1})^+$. If type i demand and type j supply were matched as much as possible, after the matching the amount of type i demand would become $\underline{x}_i \stackrel{\text{def}}{=} (\tilde{x}_i - \tilde{y}_j)^+$ and that of type j supply would become $\underline{y}_j \stackrel{\text{def}}{=} (\tilde{y}_j - \tilde{x}_i)^+$. Note that we have $\overline{x}_i - \underline{x}_i = x_i - (\tilde{y}_{j-1} - \tilde{x}_{i-1})^+ - (\tilde{x}_i - \tilde{y}_j)^+ = y_j - (\tilde{y}_{j-1} - \tilde{x}_{i-1})^- - (\tilde{x}_i - \tilde{y}_j)^- = y_j - (\tilde{x}_{i-1} - \tilde{y}_{j-1})^+ - (\tilde{y}_j - \tilde{x}_i)^+ = \overline{y}_j - \underline{y}_j$, where the second equality is due to $x_i - y_j = (\tilde{y}_{j-1} - \tilde{x}_{i-1}) + (\tilde{x}_i - \tilde{y}_j)$ and $z = z^+ - z^-$, and the third equality is due to $(-z)^- = z^+$. Thus, determining the optimal matching quantity between type i demand and type j supply is equivalent to finding the optimal protection level $a_{ij}^*(t)$, with which the post-matching levels are $u_i^* = \underline{x}_i + a_{ij}^*(t) = (\tilde{x}_i - \tilde{y}_j)^+ + a_{ij}^*(t)$ and $v_j^* = \underline{y}_j + a_{ij}^*(t) = (\tilde{y}_j - \tilde{x}_i)^+ + a_{ij}^*(t)$.

Let $a_{ij}^*(t, \tilde{x}_i - \tilde{y}_j, \mathbf{x}_{[i+1,n]}, \mathbf{y}_{[j+1,m]}) \in \arg\max_{a\geq 0} \big[-(r_i^d + r_j^s + c + h)a + \gamma E V_{t+1}(\mathbf{D}_{[1,i-1]}, \alpha(\underline{x}_i + a) + D_i, \alpha \mathbf{x}_{[i+1,n]} + \mathbf{D}_{[i+1,n]}, \mathbf{S}_{[1,j-1]}, \beta(\underline{y}_j + a) + S_j, \beta \mathbf{y}_{[j+1,m]} + \mathbf{S}_{[j+1,m]}) \big]$.

If $\overline{x}_i - \underline{x}_i = x_i - (\tilde{y}_{j-1} - \tilde{x}_{i-1})^+ - (\tilde{x}_i - \tilde{y}_j)^+ > a_{ij}^*(t)$, then it is feasible and optimal to match type i demand with type j supply until the quantity of type i demand reduces to $\underline{x}_i + a_{ij}^*(t) = (\tilde{x}_i - \tilde{y}_j)^+ + a_{ij}^*(t)$. Otherwise, it is optimal not to match type i demand with type j supply. $\square$

When we come to the decision on matching type i demand with type j supply in the top-down procedure, how much to match is determined by the optimal protection level $a_{ij}^*(t)$. If it is nonzero, the matching procedure would terminate after the matching of type i demand with type j supply; all lower quality types of demand and supply would not be matched. One managerial insight is that higher types tend to be matched in the current period to realize higher immediate reward and lower types with lower "overstocking" costs tend to be saved as safety stock for the future.

Next we consider 3 special cases of the vertical model for which we obtain more structural results.

4.3.1 Equal Carry-Over Rates

We now consider the case in which demand and supply have the *same* carry-over rate, i.e., $\alpha = \beta$, for which we can further demonstrate monotonicity properties of the optimal matching policy with respect to the system state. To proceed, we define $\tilde{D}_i \overset{\text{def}}{=} \sum_{i'=1}^{i} D_{i'}$ and $\tilde{S}_j \overset{\text{def}}{=} \sum_{j'=1}^{j} S_{j'}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. We define $\mathbf{U}_k$ as the $k \times k$ upper triangular matrix with all the entries on or above the diagonal equal to one. Then the state transformation can be written in a matrix form: $\mathbf{x}\mathbf{U}_n = \tilde{\mathbf{x}}$ and $\mathbf{y}\mathbf{U}_m = \tilde{\mathbf{y}}$. Also, let $\tilde{V}_t(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \overset{\text{def}}{=} V_t(\tilde{\mathbf{x}}\mathbf{U}_n^{-1}, \tilde{\mathbf{y}}\mathbf{U}_m^{-1}) - \tilde{\mathbf{x}}\mathbf{U}_n^{-1}(\mathbf{r}^d)^\intercal - \tilde{\mathbf{y}}\mathbf{U}_m^{-1}(\mathbf{r}^s)^\intercal$.

$\mathbf{U}_k^{-1}$ is a $k \times k$ upper-triangular difference matrix that has all diagonal entries equal to 1, $(l, l+1)$-th entry equal to -1 for all $l = 1, 2, k-1$ and all other entries equal to 0. With some algebra we can rewrite the DP (4.2) in terms of the value functions $\tilde{V}_t$ and the state variables $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ for $t = 1, \ldots, T$:

$$\begin{aligned}
\tilde{V}_t(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) &= \max_{0 \leq Q \leq \min\{\tilde{x}_n, \tilde{y}_m\}} \tilde{G}_t(Q, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}), \\
\tilde{G}_t(Q, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) &= -(1-\gamma\alpha) \sum_{i'=1}^{n} (r_{i'}^d - r_{i'+1}^d)(\tilde{x}_{i'} - Q)^+ - (1-\gamma\alpha) \sum_{j'=1}^{m} (r_{j'}^s - r_{j'+1}^s)(\tilde{y}_{j'} - Q)^+ - c(\tilde{x}_n - Q) \\
&\quad - h(\tilde{y}_m - Q) + \gamma \tilde{\mathbf{D}}\mathbf{U}_n^{-1}(\mathbf{r}^d)^\intercal + \gamma \tilde{\mathbf{S}}\mathbf{U}_m^{-1}(\mathbf{r}^s)^\intercal + \gamma E \tilde{V}_{t+1}(\alpha(\tilde{\mathbf{x}} - Q\mathbf{1}^n)^+ + \tilde{\mathbf{D}}, \alpha(\tilde{\mathbf{y}} - Q\mathbf{1}^m)^+ + \tilde{\mathbf{S}}). \qquad (4.3)
\end{aligned}$$

Lemma 4.2 *For all t, $\tilde{V}_t(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ is decreasing in $\tilde{x}_k$ for $1 \leq k < n$ and in $\tilde{y}_k$ for $1 \leq k < m$.*

Proof of Lemma 4.2. By Lemma 3.2 (set $\epsilon_t^2 = 0$), there exists $(\lambda_1, \ldots, \lambda_m) \geq 0$ such that $\sum_{j'=1}^{m} \lambda_{j'} \leq \epsilon$ and

$$\begin{aligned}
V_t(\mathbf{x}, \mathbf{y}) - V_t(\mathbf{x} + \epsilon\mathbf{e}_i^n - \epsilon\mathbf{e}_{i'}^n, \mathbf{y}) =& V_t((\mathbf{x} + \epsilon\mathbf{e}_i^n - \epsilon\mathbf{e}_{i'}^n) - \epsilon\mathbf{e}_i^n + \epsilon\mathbf{e}_{i'}^n, \mathbf{y}) - V_t(\mathbf{x} + \epsilon\mathbf{e}_i^n - \epsilon\mathbf{e}_{i'}^n, \mathbf{y}) \\
&\geq -\sum_{j'=1}^{m} \lambda_{j'}(r_{ij'} - r_{i'j'}) = -\sum_{j'=1}^{m} \lambda_{j'}(r_i^d - r_{i'}^d).
\end{aligned}$$

If $i < i'$, then $r_i^d > r_{i'}^d$ and $V_t(\mathbf{x},\mathbf{y}) - V_t(\mathbf{x}+\epsilon\mathbf{e}_i^n - \epsilon\mathbf{e}_{i'}^n,\mathbf{y}) \ge -\sum_{j'=1}^m \lambda_{j'}(r_i^d - r_{i'}^d) \ge -(r_i^d - r_{i'}^d)\epsilon$. Then, for $1 \le k < n$, we have

$$\begin{aligned}
&\tilde{V}_t(\tilde{\mathbf{x}}+\epsilon\mathbf{e}_k^n,\tilde{\mathbf{y}}) - \tilde{V}_t(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\\
=&V_t((\tilde{\mathbf{x}}+\epsilon\mathbf{e}_k^n)\mathbf{U}_n^{-1},\tilde{\mathbf{y}}\mathbf{U}_m^{-1}) - V_t(\tilde{\mathbf{x}}\mathbf{U}_n^{-1},\tilde{\mathbf{y}}\mathbf{U}_m^{-1}) - (\tilde{\mathbf{x}}+\epsilon\mathbf{e}_k^n)\mathbf{U}_n^{-1}(\mathbf{r}^d)^\intercal - \tilde{\mathbf{x}}\mathbf{U}_n^{-1}(\mathbf{r}^d)^\intercal\\
=&V_t(\mathbf{x}+\epsilon\mathbf{e}_k^n - \epsilon\mathbf{e}_{k+1}^n,\mathbf{y}) - V_t(\mathbf{x},\mathbf{y}) - (r_k^d - r_{k+1}^d)\epsilon \le 0.
\end{aligned}$$

Thus, $\tilde{V}_t(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ is decreasing in $\tilde{x}_k$ for $1 \le k < n$. Similarly, $\tilde{V}_t(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ is decreasing in $\tilde{y}_k$ for $1 \le k < m$. □

To proceed further on the monotonicity properties of the optimal matching policy, we make use of the notion of $L^\natural$-concavity. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called $L^\natural$-convex if $f(\mathbf{x} - \xi\mathbf{1}^n)$ is submodular in $(\mathbf{x},\xi)$ (see Murota 2003). A function g is $L^\natural$-concave if $-g$ is $L^\natural$-convex.

Lemma 4.3 *Suppose $\alpha = \beta$. $\tilde{V}_t(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ is $L^\natural$-concave in $(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ for $t = 1,\dots,T+1$, and $\tilde{G}_t(Q,\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ is $L^\natural$-concave in $(Q,\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ for $t = 1,\dots,T$.*

Proof of Lemma 4.3. The proof is by induction on t. Clearly, $\tilde{V}_{T+1}(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) \equiv 0$ is $L^\natural$-concave in $(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$. We suppose that $\tilde{V}_{t+1}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ is $L^\natural$-concave in $(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$. Then by definition of $L^\natural$-concavity and submodularity, for any given $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{S}}$, $\tilde{V}_{t+1}(\alpha\tilde{\mathbf{x}}+\tilde{\mathbf{D}},\alpha\tilde{\mathbf{y}}+\tilde{\mathbf{S}})$ is $L^\natural$-concave in $(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$. Now consider period t. Since $Q \le \min\{\tilde{x}_n,\tilde{y}_m\}$ and $\alpha = \beta$, We have that $\tilde{V}_{t+1}(\alpha(\tilde{\mathbf{x}}-Q\mathbf{1}^n)^+ + \tilde{\mathbf{D}},\alpha(\tilde{\mathbf{y}}-Q\mathbf{1}^m)^+ + \tilde{\mathbf{S}}) = \tilde{V}_{t+1}(\alpha(\tilde{\mathbf{x}}_{[1,n-1]} - Q\mathbf{1}^{n-1})^+ + \tilde{\mathbf{D}}_{[1,n-1]},\alpha(\tilde{x}_n - Q) + \tilde{D}_n,\alpha(\tilde{\mathbf{y}}_{[1,m-1]} - Q\mathbf{1}^{m-1})^+ + \tilde{\mathbf{S}}_{[1,m-1]},\alpha(\tilde{y}_m - Q) + \tilde{S}_m)$ is $L^\natural$-concave in $(Q,\tilde{\mathbf{x}},\tilde{\mathbf{y}})$, by applying Chen et al. (2014, Lemma 4) and noting the monotonicity proved in Lemma 4.2. By Simchi-Levi et al. (2014, Proposition 2.3.4(c)), $E_{\tilde{\mathbf{D}},\tilde{\mathbf{S}}}[\tilde{V}_{t+1}(\alpha(\tilde{\mathbf{x}}-Q\mathbf{1}^n)^+ + \tilde{\mathbf{D}},\alpha(\tilde{\mathbf{y}} - Q\mathbf{1}^m)^+ + \tilde{\mathbf{S}})]$ is $L^\natural$-concave in $(Q,\tilde{\mathbf{x}},\tilde{\mathbf{y}})$, thus the last term in (4.3) is $L^\natural$-concave in $(Q,\tilde{\mathbf{x}},\tilde{\mathbf{y}})$. The first two terms in (4.3) are $L^\natural$-concave in $(Q,\tilde{\mathbf{x}},\tilde{\mathbf{y}})$, because $-(\tilde{x}_{i'} - Q)^+$ is supermodular in $(Q,\tilde{x}_{i'})$, $-(\tilde{y}_{j'} - Q)^+$ is supermodular in $(Q,\tilde{y}_{j'})$ and $L^\natural$-concavity is preserved under any nonnegative linear combination. Since the other terms are linear, $\tilde{G}_t(Q,\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ is $L^\natural$-concave in $(Q,\tilde{\mathbf{x}},\tilde{\mathbf{y}})$. By Simchi-Levi et al. (2014, Proposition 2.3.4(e)), $\tilde{V}_t(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ is $L^\natural$-concave in $(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$. This completes the induction. □

With the value functions in the transformed system proven to be $L^\natural$-concave, we obtain the following monotonicity properties of the optimal matching policy for the original system.

Theorem 4.5 (Vertical model: monotonicity property of optimal total matching quantity) *Suppose $\alpha = \beta$. The optimal total matching quantity $Q_t^*(\mathbf{x},\mathbf{y})$ is nondecreasing in $(\mathbf{x},\mathbf{y})$ and satisfies that, for $\epsilon > 0$,*

(i) $Q_t^*(\mathbf{x}+\epsilon\mathbf{e}_1^n,\mathbf{y}+\epsilon\mathbf{e}_1^m) = Q_t^*(\mathbf{x},\mathbf{y}) + \epsilon$,

(ii) $0 \le Q_t^*(\mathbf{x}+\epsilon\mathbf{e}_n^n,\mathbf{y}) - Q_t^*(\mathbf{x},\mathbf{y}) \le Q_t^*(\mathbf{x}+\epsilon\mathbf{e}_{n-1}^n,\mathbf{y}) - Q_t^*(\mathbf{x},\mathbf{y}) \le \cdots \le Q_t^*(\mathbf{x}+\epsilon\mathbf{e}_1^n,\mathbf{y}) - Q_t^*(\mathbf{x},\mathbf{y}) \le \epsilon$,

(iii) $0 \le Q_t^*(\mathbf{x},\mathbf{y}+\epsilon\mathbf{e}_m^m) - Q_t^*(\mathbf{x},\mathbf{y}) \le Q_t^*(\mathbf{x},\mathbf{y}+\epsilon\mathbf{e}_{m-1}^m) - Q_t^*(\mathbf{x},\mathbf{y}) \le \cdots \le Q_t^*(\mathbf{x},\mathbf{y}+\epsilon\mathbf{e}_1^m) - Q_t^*(\mathbf{x},\mathbf{y}) \le \epsilon$.

Proof of Theorem 4.5. Monotonicity of $Q_t^*(\mathbf{x},\mathbf{y})$. Since $L^\natural$-concavity implies supermodularity, by Lemma 4.3, $\tilde{G}_t(Q,\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ is $L^\natural$-concave, a fortiori, supermodular in $(Q,\tilde{\mathbf{x}},\tilde{\mathbf{y}})$. By Simchi-Levi et al. (2014, Theorem 2.2.8), the optimal solution to (4.3), denoted by $\hat{Q}_t(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$, is nondecreasing in $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$. Since the higher the original state $(\mathbf{x}, \mathbf{y})$, the higher the transformed state $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$, the optimal solution $Q_t^*(\mathbf{x}, \mathbf{y})$, expressed in terms of the original state, is nondecreasing in $(\mathbf{x}, \mathbf{y})$.

(i) Note that $(\mathbf{x} + \epsilon \mathbf{e}_1^n, \mathbf{y} + \epsilon \mathbf{e}_1^m)$ is a state that has ϵ more type 1 demand and supply than state $(\mathbf{x}, \mathbf{y})$, and it is optimal to match type 1 demand and supply as much as possible; thus after the first round of matching type 1 demand and supply, there are the same levels of the remaining types for the system with state $(\mathbf{x} + \epsilon \mathbf{e}_1^n, \mathbf{y} + \epsilon \mathbf{e}_1^m)$ and with state $(\mathbf{x}, \mathbf{y})$. Thus the optimal matching decisions for the remaining types must be the same for the two states, and as a result, $Q_t^*(\mathbf{x}+\epsilon \mathbf{e}_1^n, \mathbf{y}+\epsilon \mathbf{e}_1^m) = Q_t^*(\mathbf{x}, \mathbf{y})+\epsilon$.

(ii) By the definition of $L^\natural$-concavity, $\tilde{G}_t(Q - \xi, \tilde{\mathbf{x}} - \xi \mathbf{1}^n, \tilde{\mathbf{y}} - \xi \mathbf{1}^m)$ is supermodular in $(Q, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}, \xi)$. Then, for $Q > \hat{Q}_t(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) + \epsilon$, we have

$$\tilde{G}_t(Q, \tilde{\mathbf{x}} + \epsilon \mathbf{1}^n, \tilde{\mathbf{y}} + \epsilon \mathbf{1}^m) - \tilde{G}_t(\hat{Q}_t(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) + \epsilon, \tilde{\mathbf{x}} + \epsilon \mathbf{1}^n, \tilde{\mathbf{y}} + \epsilon \mathbf{1}^m) \leq \tilde{G}_t(Q - \epsilon, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) - \tilde{G}_t(\hat{Q}_t(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}), \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \leq 0,$$

where the first inequality is derived by definition of supermodularity and the second inequality is due to the optimality of $\hat{Q}_t$. This implies that any matching quantity $Q > \hat{Q}_t(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) + \epsilon$ is no better than $\hat{Q}_t(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) + \epsilon$ for the state $(\tilde{\mathbf{x}} + \epsilon \mathbf{1}^n, \tilde{\mathbf{y}} + \epsilon \mathbf{1}^m)$. Therefore, $\hat{Q}_t(\tilde{\mathbf{x}} + \epsilon \mathbf{1}^n, \tilde{\mathbf{y}} + \epsilon \mathbf{1}^m) \leq \hat{Q}_t(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) + \epsilon$. By the monotonicity of $\hat{Q}_t(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$, $\hat{Q}_t(\tilde{\mathbf{x}}+\epsilon \mathbf{1}^n, \tilde{\mathbf{y}}) \leq \hat{Q}_t(\tilde{\mathbf{x}}+\epsilon \mathbf{1}^n, \tilde{\mathbf{y}}+\epsilon \mathbf{1}^m) \leq \hat{Q}_t(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})+\epsilon$. Expressed in the original state, $Q_t^*(\mathbf{x} + \epsilon \mathbf{e}_1^n, \mathbf{y}) \leq Q_t^*(\mathbf{x}, \mathbf{y}) + \epsilon$, which proves the last inequality in part (ii).

For any two original states $(\mathbf{x} + \epsilon \mathbf{e}_k^n, \mathbf{y})$ and $(\mathbf{x} + \epsilon \mathbf{e}_{k+1}^n, \mathbf{y})$, $k = 1, \ldots, n-1$, their transformed states can be ordered as $(\tilde{\mathbf{x}} + \epsilon \mathbf{1}_{[k,n]}, \tilde{\mathbf{y}}) \geq (\tilde{\mathbf{x}} + \epsilon \mathbf{1}_{[k+1,n]}, \tilde{\mathbf{y}})$, where $\mathbf{1}_{[k,n]}$ is an n-dimensional vector with the k-th up to n-th entry being one and the rest of the entries being all zeros. By the monotonicity of $\hat{Q}_t(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$, $\hat{Q}_t(\tilde{\mathbf{x}} + \epsilon \mathbf{1}_{[k,n]}, \tilde{\mathbf{y}}) \geq \hat{Q}_t(\tilde{\mathbf{x}} + \epsilon \mathbf{1}_{[k+1,n]}, \tilde{\mathbf{y}})$. Translated into the original state, $Q_t^*(\mathbf{x} + \epsilon \mathbf{e}_k^n, \mathbf{y}) \geq Q_t^*(\mathbf{x} + \epsilon \mathbf{e}_{k+1}^n, \mathbf{y})$ and thus, $Q_t^*(\mathbf{x} + \epsilon \mathbf{e}_{k+1}^n, \mathbf{y}) - Q_t^*(\mathbf{x}, \mathbf{y}) \leq Q_t^*(\mathbf{x} + \epsilon \mathbf{e}_k^n, \mathbf{y}) - Q_t^*(\mathbf{x}, \mathbf{y})$. Combining that with $Q_t^*(\mathbf{x} + \epsilon \mathbf{e}_1^n, \mathbf{y}) \leq Q_t^*(\mathbf{x}, \mathbf{y}) + \epsilon$, we have the desired series of inequalities in part (ii), with the first inequality implied by the monotonicity of $Q_t^*(\mathbf{x}, \mathbf{y})$.

(iii) The series of inequalities can be proved analogously to part (ii). □

Theorem 4.5 provides first-order monotonicity properties of the optimal total matching quantity with respect to the state for vertically differentiated types. First, the higher the levels of demand and supply, the more quantities are optimally matched in a period. Second, part (i) is a direct consequence of Theorem 3.2. It says if the levels of type 1 demand and supply are increased by the same amount, this increased amount will be optimally matched between them in the current period. Third, the series of inequalities (i.e., in parts (ii) and (iii)) show that an increment in the level of a demand or supply type with higher "quality" leads to a higher optimal matching quantity, and the rate of increase is dominated by 1. The statement is consistent with the intuition that higher types are more likely to be matched in the current period. We caution that these results are obtained under the assumption of equal carry-over rates; i.e., $\alpha = \beta$. This is because these monotonicity properties are built upon the $L^\natural$-concavity of the value functions in the transformed system. Unlike concavity and supermodularity, $L^\natural$-concavity depends on the scaling of the variables (Zipkin 2008). (One may expect similar properties for unequal carry-over rates, which may call for a novel form of concavity. We leave that to future research.)

The following corollary recounts Theorem 4.5 in terms of the state-dependent protection levels.

Corollary 4.3 (Vertical model: monotonicity property of optimal protection level) *Suppose* $\alpha = \beta$. *The state-dependent protection level* $a^*_{ij}(t, \tilde{x}_i - \tilde{y}_j, \mathbf{x}_{[i+1,n]}, \mathbf{y}_{[j+1,m]})$ *is nonincreasing in* $(\tilde{x}_i - \tilde{y}_j)^+$, $(\tilde{x}_i - \tilde{y}_j)^-$, $\mathbf{x}_{[i+1,n]}$ *and* $\mathbf{y}_{[j+1,m]}$, *with the decreasing rates no more than 1. In particular,* $a^*_{11}(t) \equiv 0$. *Moreover,* $a^*_{ij}(t)$ *is most sensitive to* $\tilde{x}_i - \tilde{y}_j$ *and is more sensitive to* $x_{i'}$ *than to* $x_{i'+1}$ *and to* $y_{j'}$ *than to* $y_{j'+1}$ *for* $i+1 \le i' \le n-1$ *and* $j+1 \le j' \le m-1$.

4.3.2 Lost Demand or Supply

When $\beta = 0$, any unmatched supply does not carry over to the next period. Similarly, $\alpha = 0$ means that unmatched demand will be lost. By symmetry, we focus on the case in which $\beta = 0$.

Proposition 4.2 (Vertical model: lost supply) *With a stronger assumption* $\beta = 0$, *Theorem 4.4 can be strengthened as follows: In considering the matching of type* i *demand with type* j *supply, there exists a state-dependent threshold* $\theta_{ij}(t, \mathbf{x}_{[i+1,n]})$ *such that it is optimal to reduce type* i *demand to* $\theta_{ij}(t, \mathbf{x}_{[i+1,n]})$ *if* $\tilde{x}_i - \tilde{y}_j < \theta_{ij}(t, \mathbf{x}_{[i+1,n]}) < \tilde{x}_i - \tilde{y}_{j-1}$, *to match it with type* j *supply down to the level* $\tilde{x}_i - \tilde{y}_j$ *if* $\tilde{x}_i - \tilde{y}_j \ge \theta_{ij}(t, \mathbf{x}_{[i+1,n]})$ *and otherwise not to match type* i *demand and type* j *supply.*

The proof is straightforward and omitted. Due to lost supply, the threshold $\theta_{ij}(t, \mathbf{x}_{[i+1,n]})$ that determines the matching between type i demand and type j supply has a lower-dimensional state dependency, only depending on the time and states of all demand types of lower quality than the focal type i.

4.3.3 1 Demand Type and m Supply Types

We consider the model with only 1 demand type and m supply types. The next result immediately follows from Theorem 4.4 (we omit its proof since it is straightforward).

Corollary 4.4 (1-to-m vertical model) *With a stronger assumption of 1 demand type and* m *supply types, Theorem 4.4 can be strengthened as follows: In considering the matching type 1 demand with type* j *supply, there exists a threshold* $\bar{z}_j(t, x_1 - \tilde{y}_j, \mathbf{y}_{[j+1,m]})$ *such that it is optimal to reduce the type 1 demand to* $\min\{\bar{z}_j(t, x_1 - \tilde{y}_j, \mathbf{y}_{[j+1,m]}), x_1 - \tilde{y}_{j-1}\}$ *by matching it with type* j *supply.*

In the vertical 1-to-m model, the single demand type is matched with supply types sequentially from high quality to low quality. In considering its matching with type j supply, the remaining demand level is $x_1 - \tilde{y}_{j-1}$. There exists an optimal match-down-to level $\bar{z}_j(t, x_1 - \tilde{y}_j, \mathbf{y}_{[j+1,m]})$ such that it is optimal to match the demand down to that level if the available demand is more than the level, and otherwise not to match type 1 demand and type j supply as well as all the lower-quality supply types. Furthermore, we provide conditions under which the optimal match-down-to levels become state-independent, which can be computationally desirable.

Proposition 4.3 *The optimal match-down-to level* $\bar{z}_j(t, x_1 - \tilde{y}_j, \mathbf{y}_{[j+1,m]})$ *in the 1-to-m vertical model becomes a constant, which is dependent on* t *but independent of* $x_1 - \tilde{y}_j$ *and* $\mathbf{y}_{[j+1,m]}$*, if* $\beta = 0$ *or* $\alpha = \beta$*.*

Proof of Proposition 4.3. The result for $\beta = 0$ follows directly from Proposition 4.2. For $\alpha = \beta$, the result can be proved by applying the same approach as in Yu et al. (2015), thus we omit the details. □

We can obtain analogous results for the vertical model with n demand types and 1 supply type.

4.4 Conclusion

In this chapter, we study the dynamic type matching problem introduced in Chapter 3 with two reward structures that satisfy the modified Monge condition for all neighboring pairs. For both reward structures, we characterize the structure of the optimal matching policy. In the unidirectionally horizontal reward structure, "distance" determines priority, and in the vertical reward structure, "quality" determines priority. Under those two specialized reward structures, along the priority hierarchy, when it comes to the matching between a specific pair, the optimal policy has a match-down-to threshold structure. This structure connects back to the base-stock policy in inventory management and the protection-level policy in quantity-based revenue management.

In addition to the above two reward structures, our results can also be applied to other forms of reward structures to partially characterize the optimal matching policy in those settings. For example, in a horizontal circle model, if we consider the shortest distance rather than the unidirectional distance, a parallel version of Theorem 4.1 can be established. Suppose that r_{ij} is a linearly nonincreasing function of the shortest distance. Then, we can verify by definition that $(i, j) \succeq (i', j')$, if (i) the shortest path from i to j on the circle (i.e., the shorter circular segment between i and j) is a subset of the shortest path from i' to j' and (ii) the shortest-distance travel from j to i is in the same direction as the shortest-distance travel from j' to i'. However, the two types $i \in \mathcal{D}$ and $j \in \mathcal{S}$ that are closest to each other on the circle may not constitute a perfect pair anymore. In the vertical model, it is again verifiable by definition that if $r_i^d \geq r_j^s \geq r_{j'}^s$ or $r_i^d \leq r_j^s \leq r_{j'}^s$, then $(i, j) \succeq (i, j')$ when $r_{ij} = \min\{r_i^d, r_j^s\}$, and $(i, j') \succeq (i, j)$ when $r_{ij} = \max\{r_i^d, r_j^s\}$.

Though those two reward structures include many classic and emerging problems as special cases, many practical settings generally fail to satisfy the modified Monge condition. For example, in the vertical model, if the reward function is a general supermodular function other than an additive one, there exist scenarios in which socially efficient matching is not a top-down matching (i.e., assortative mating). In the horizontal model, if the distance is the shortest distance, there exist scenarios in which "matching-to-the-closest" greedily is not optimal. Characterizing the optimal matching policy for those reward structures will be interesting but challenging. We leave that for future research.

Chapter 5

Pricing Behaviors under Social Incentives

5.1 Introduction

In the sharing economy, the intermediary platforms often use independent suppliers to provide service to customers. Naturally, those independent suppliers are involved in competition with each other in some way for customers. For example, the hosts on AirBnB compete with each other through pricing strategies and quality of service. Research in psychology, sociology, and economics suggests that social comparison has become increasingly important for understanding competitive behaviors of individuals (Bellemare et al. 2008; Fehr and KM 1999; Iyer and Soberman 2016; Kahneman and Tversky 1979; Kőszegi and Rabin 2006; Loewenstein et al. 1989). Individuals involved in sharing economy activities, consumers, and even managers of firms in various industries, as human beings, naturally practice social comparison. In addition, within a firm, social comparison behavior between agents can also be induced or enforced internally by the firms by providing certain forms of incentives. For example, executive bonuses or sales compensations often depend on how much a manager's performance exceeds his/her peers' or the industry average (Cui et al. 2015; Ho and Su 2009; Lim 2010; Lim et al. 2009; Main et al. 1993). Inter-firm social comparison also exists, due to incentives from outside the firms, e.g., media comparisons, third-party ranking, and most importantly, reception by investors. A top-performing firm typically attracts investors with a large flow of capital, whereas a bottom performer can be severely punished by the capital market.

There at least exists two forms social incentives that lead to comparison: *behind aversion* (or upward social comparison), i.e., a feeling of loss when getting a worse outcome than other consumers or competitors, and *ahead seeking* (or downward social comparison), i.e., a sense of achievement from obtaining a better outcome than others (Amaldoss and Jain 2008, 2010, 2015; Kuksov and Xie 2012).[1] Social comparison theory implies that *enhanced competitiveness* is a natural outcome of the social comparison process (Garcia et al. 2013), regardless of the direction of social comparisons and the relative positions of the decision makers. On the one hand, upward comparison by a person in an inferior position can lead him/her to behave competitively in order to reduce discrepancies between him/her and the person above. On the other hand, downward comparison by a person in a superior position can also lead to competitive behavior in order to maintain his/her position, which may be threatened by an upward comparison from below. Indeed, we confirm this theory in the canonical duopoly price competition of differentiated substitutable products in a deterministic environment (Chen and Cui 2013; Narasimhan 1988; Singh and Vives 1984).

Today's economy is no doubt filled with uncertainty. Can the insights obtained for a deterministic environment be applicable to uncertain environments? In contrast to the existing social comparison theory, we show how demand uncertainty changes the competitive landscape in the presence of social comparison, where upward and downward comparisons play different roles. In particular, we consider classic duopoly price competition between two agents in a random market environment. We show that the more behind-averse managers are, the more intense price competition will be. Somewhat surprisingly, the influence of ahead-seeking behavior by agents on profitability depends on the market variability: more prominent ahead seeking behavior would reduce price competition if the market variability is large enough but otherwise would intensify price competition.

Those results are due to the interplay of two effects – an *expected*-comparison effect and a *variable*-inequality effect. Like social comparison theory, the expected-comparison effect captures the influence of social comparison on competitive pricing decisions regardless of demand variability. As in a deterministic environment, social comparison behavior, whether behind aversion or ahead seeking, always intensifies price competition. In addition, the variable-inequality effect captures the effect of interactions between social comparison and demand variability on competitive pricing decisions. Note that the agents commit to a price ex ante subject to market variability. When ahead-seeking behavior becomes more prominent, in deciding the committed price agents put more weights on those higher market realizations. Since a bigger market warrants a higher price, the larger weight on more booming markets reduces price competition. On the other hand, when behind aversion becomes more significant, agents put more weights on those lower market realizations, thereby intensifying price competition. If the market variability is large enough, the anti-competitive variable-inequality effect, which is induced by more prominent ahead-seeking behavior, may dominate the pro-competitive expected-comparison effect. Hence, counter-intuitively, ahead seeking leads to an overall anti-competitive outcome in a highly variable market. On the other hand, behind aversion always leads to an overall pro-competitive outcome, regardless of demand variability. It is such an interplay of both effects that leads to the counter-intuitive

[1] Another form is *distributive fairness*, i.e., a sense of sympathy for competitors when surpassing them (Fehr and KM 1999). Our model also accounts for this form of behavior.

results in an environment with demand variability than without variability.

In addition, we also find that agents' biased perceptions of market variability may also affect price competition in an interesting way. More specifically, if managers overestimate the market variability facing their competitors, such a biased belief can reduce price competition. That is because the variable-inequality effect is influenced in different ways by the agent's own market variability and by the apparent market variability of the competitor.

Figure 5.1: The Moderating Role of Demand Variability

	Substitution	**Complementarity**
Behind Aversion	↓	↑ for low variability ↓ for high variability
Ahead Seeking	↓ for low variability ↑ for high variability	↑

Note: ↑: anti-competitive; ↓: pro-competitive.

We also make several extensions. First, all results obtained for the additive demand uncertainty are preserved under the multiplicative demand uncertainty, demonstrating the robustness of the managerial insights obtained. Second, we show that the main results obtained for substitutable products are *reversed* for complementary products (See Figure 5.1 for a summary). With complementary products, the more ahead-seeking agents are, the less intense price competition will be. On the other hand, behind aversion may also reduce price competition if the variability of the market is small enough; otherwise, it will intensify price competition. Moreover, whereas market variability lowers equilibrium prices and reduces agents' profitability in the case of substitutable products, surprisingly, there is an inverted U-shape relationship between agents' profitability and market variability in the case of complementary products. In other words, agents selling complementary products can benefit from demand variability as long as it is not too high. Lastly, we show that the insights obtained for the linear demand structure are robust for non-linear demand curves.

5.2 Literature Review

Price competition is one of core themes in firms' strategic interactions in marketing (see, e.g., Chen et al. 2001; Iyer 1998; Iyer et al. 2005; Narasimhan 1988; Vives 1999, among others). Narasimhan (1988), for example, studies the equilibrium duopoly pricing strategies when the firms compete for both loyal consumers and switchers. Iyer (1998) examines how a manufacturer coordinates distribution channels when retailers are engaged in both price and non-price competition for end consumers. Our work differs from those papers by incorporating social comparison in models of duopoly price competition.

To the best of our knowledge, our work is the first to study how demand variability affects price competition with social comparison. Nor are we aware of any analytical work on the influence of decision makers' social comparisons on their *strategic interactions* (with the underlying assumption being that they interact with one another even without social comparisons). In a recent and significant contribution to the literature of social comparison, Iyer and Soberman (2016) examine how social comparison between socially responsible consumers may influence firms' product innovation strategies. Lim (2010) studies how a principal would design a sales contest when sales agents care about their contest outcomes relative to other contestants, and finds that agents' social comparison induces the principal to offer a higher proportion of winners than without social comparison. Our work differs from those papers by studying how social comparison may influence firms' strategic price decisions when there is demand variability.

The managerial insights of our work may also shed light on other competition in the presence of social incentives (see Section 5.6). For substitutable products, pricing decisions are strategic complements, whereas for complementary products they are strategic substitutes. We show that the effects of social incentives on competitive behavior for substitutable products are reversed for complementary products.

Social comparisons are also studied in other domains, such as social psychology and behavioral economics. See a recent comprehensive literature review in Roels and Su (2014). We confirm the empirical and experimental evidence that social comparison leads to more competitive behavior in a deterministic environment (see, e.g., Falk and Ichino 2006; Mas and Moretti 2009). Yet we also identify novel effects of social comparisons in a random environment. The theoretical results provide a testable hypothesis (i.e., in a random environment, behind aversion intensifies competition; but ahead seeking may alleviate competition when the degree of uncertainty is high) for decision making in the presence of social comparison in an uncertain environment, a hypothesis which may have been overlooked so far.

A growing number of papers in the marketing and operations management literature also consider social preferences of decision makers, contributing to the large literature of social preferences in economics (Anderson and Simester 2008; Chen and Cui 2013; Cui and Mallcucci 2016; Fehr et al. 2007; Fehr and KM 1999; Ho and Su 2009; also see Goldfarb et al. 2012 for a survey that calls for studies on behavioral models of managerial decision-making). Loch and Wu (2008) provide experimental evidence that social preferences, such as status seeking and reciprocation, systematically affect firms' economic decisions. Ho et al. (2014) consider a one-supplier and two-retailer distribution channel and study how distributional and peer-induced fairness might influence the design of wholesale price contracts.

Some scholars examine stochastic reference points in decision making under uncertainty. Ho et al. (2010) consider managers' stochastic reference dependence behavior when studying a multi-location newsvendor problem, with the realized demands as the reference points. The authors provide a theoretical explanation to the pull-to-center bias observed in earlier experiments. Avcı et al. (2014) study managers' behind aversion and ahead-seeking behavior in making ordering decisions in a competitive newsvendor setting, with stochastic reference points of the possible competitors' profit outcomes. In Avcı et al. (2014), decision makers do *not* interact strategically with one another if there are no social comparisons. In contrast, we focus on how social comparisons interact with demand variability in influencing firms' competitive pricing decisions.

In a related emerging stream of research, a handful of papers study loss aversion with deterministic or stochastic reference points of *consumers*, and the firm's strategies in response. With deterministic consumer reference points, Popescu and Wu (2007) study a discrete-time infinite-horizon monopolistic pricing problem under a general nonlinear reference-dependent demand model. Nasiry and Popescu (2011) study a version with the reference point as a weighted average of the lowest and most recent prices. Nasiry and Popescu (2012) characterize the effect of anticipated regret on consumer decisions and on firms' profits and policies in an advance selling context where buyers have uncertain valuations. Liu and Shum (2013) study a firm's optimal pricing and rationing decisions over two periods in anticipation of possible consumer disappointment due to stockouts. Özer and Zheng (2015) study a seller's optimal pricing and inventory strategies when anticipated regret and misperception of product availability affect consumers' purchase decisions. Yang et al. (2014a) consider service systems competitively setting average wait times or prices when consumers exhibit loss aversion behavior against deterministic benchmarks. With stochastic consumer reference points, three papers apply the framework of endogenized reference points as personal equilibrium, proposed by Kőszegi and Rabin (2006), to model consumers' loss aversion behavior in an operations context. Baron et al. (2015) consider a repeated newsvendor setting, Yang et al. (2014b) consider a service system system, and Courty and Nasiry (2014) focus on quality-dependent consumer valuations.

5.3 Model Setup

Consider a symmetric duopoly of agents 1 and 2 facing random price-sensitive demand. The demand function of agent i, $i = 1, 2$, denoted by $D_i(p_1, p_2, \epsilon_i)$, is in the following form:

$$D_i(p_1, p_2, \epsilon_i) = d_i(p_1, p_2) + \epsilon_i = m - p_i + \gamma p_{-i} + \epsilon_i, \tag{5.1}$$

where $-i$ denotes the competitor of agent i, and ϵ_i is a random variable as an *additive* shock to the potential market size m. (We consider the *multiplicative* shock in Section 5.5.1.) Without loss of generality, let $E\epsilon_1 = E\epsilon_2 = 0$. Then, it is clear that the expected demands of the two agents are $d_1(p_1, p_2)$ and $d_2(p_1, p_2)$, respectively. In addition, we assume that ϵ_1 and ϵ_2 follow independent and identically distributed (i.i.d.), *symmetric* probability distributions as ϵ on $[-\bar{\alpha}, \bar{\alpha}]$. In other words, there are symmetrically distributed upswings and downswings in the market size.

We first consider the case that the cross-product price sensitivity satisfies $0 \le \gamma < 1$. In other words, the two agents sell *substitutable* products, and the pricing decision of one agent has less effect on the other agent's demand than on her own demand. (We consider the case of complementary products, i.e., $-1 < \gamma < 0$, in Section 5.5.2.)

Without loss of generality, we assume that the marginal cost is $c = 0$ per unit.[2] Then, given a random shock ϵ_i, the profit of agent i, $i = 1, 2$, is given by:

$$\Pi_i(p_1, p_2, \epsilon_i) = p_i D_i(p_1, p_2, \epsilon_i) = p_i \left[d_i(p_1, p_2) + \epsilon_i\right].$$

In addition to their profits, a gain-loss utility is incurred for both agents when the profits are unequal. We write such utility under social comparisons for agents as follows:

$$\begin{aligned} S_i(p_1, p_2, \boldsymbol{\epsilon}) =& e\left[\Pi_i(p_1, p_2, \epsilon_i) - \Pi_{-i}(p_1, p_2, \epsilon_{-i})\right]^+ - \ell\left[\Pi_i(p_1, p_2, \epsilon_i) - \Pi_{-i}(p_1, p_2, \epsilon_{-i})\right]^- \\ =& e[\Pi_i(p_1, p_2, \epsilon_i) - \Pi_{-i}(p_1, p_2, \epsilon_{-i})] - (\ell - e)\left[\Pi_i(p_1, p_2, \epsilon_i) - \Pi_{-i}(p_1, p_2, \epsilon_{-i})\right]^-, \end{aligned} \tag{5.2}$$

where $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2)$, $x^+ = \max\{x, 0\}$ and $x^- = -\min\{x, 0\}$. The second equality is due to $x = x^+ - x^-$. The agents are *behind-averse*, namely, one agent dislikes the situation in which the other agent earns a higher profit than herself (i.e., $\ell \geq 0$). The agents are also *ahead-seeking*, namely, the agent prefers to outperform her peer (i.e., $e \geq 0$). In view of the behavioral literature (e.g., the celebrated prospect theory, see Kahneman and Tversky 1979), we require $0 \leq e \leq \ell$, which means that the loss due to an underperformance weighs more than equal-sized gain due to an overperformance.[3]

For a pair of demand shocks $\boldsymbol{\epsilon}$, the total utility of agent i, $i = 1, 2$, is the sum of its profit and its gain-loss utility under social comparison; i.e., $U_i(p_1, p_2, \boldsymbol{\epsilon}) = \Pi_i(p_1, p_2, \epsilon_i) + S_i(p_1, p_2, \boldsymbol{\epsilon})$. We confine the price p_i, $i = 1, 2$, to the interval $[0, p^{\max}]$.

Assumption 5.1 *(*Regular Price Range*). The upper bound, $p^{\max}$, on the price range satisfies: (i) if $\gamma \geq 0$, $p^{\max} \leq m - \bar{\alpha}$, or (ii) if $\gamma < 0$, $p^{\max} \leq \frac{m-\bar{\alpha}}{1-\gamma}$.*

Assumption 5.1 ensures that the ex post demand $d_i(p_1, p_2) + \epsilon_i$ is always nonnegative, regardless of the realization of ϵ_i for any ex-ante pair of price choices $(p_1, p_2) \in [0, p^{\max}]^2$. Even though all the subsequent results hold without Assumption 5.1, it is only natural that any realization of demand is nonnegative.

Agents simultaneously set their prices before the random market shocks realize, with agent i's objective being to maximize its expected total utility, i.e., to solve $\max_{p_i \in [0, p^{\max}]} u_i(p_1, p_2) \equiv E_{\boldsymbol{\epsilon}}[U_i(p_1, p_2, \boldsymbol{\epsilon})]$. The following lemma shows that the ex post utility functions are well-behaved.

Lemma 5.1 *For any $\boldsymbol{\epsilon}$, $U_i(p_i, p_{-i}, \boldsymbol{\epsilon})$, $i = 1, 2$, is concave in p_i for a given p_{-i}.*

Proof of Lemma 5.1. We prove that $U_1(p_1, p_2, \boldsymbol{\epsilon})$ is concave in p_1 for a given p_2. The proof for $U_2(p_1, p_2, \boldsymbol{\epsilon})$ being concave in p_2 is exactly the same due to symmetry. In order to do this, we first show

[2] For positive supply cost $c > 0$, the model can be converted to the same form as the case with $c = 0$ by defining $p_i^c = p_i - c$ for $i = 1, 2$ as the adjusted price. The analysis for $c > 0$ in the presence of revenue comparisons is analogous.

[3] Our model can also allow the agents to be *distributively fair*, namely, the agent does not prefer to earn a higher profit than its peer because of fairness concerns, i.e., $e < 0$. For the distributive fairness case, we also require that $e \geq -1$, under which the disutility due to fairness concerns would not completely erode the profit gain itself. Though our results apply to both distributive fairness and ahead seeking, for the sake of simplicity, we restrict ourselves to ahead seeking in the base model. See Section 5.6 for more discussions.

that $\Pi_1(p_1,p_2,\epsilon_1)-\Pi_2(p_1,p_2,\epsilon_2)$ is concave in p_1. Let us investigate its differential with respect to p_1.

$$\frac{\partial\left[\Pi_1(p_1,p_2,\epsilon_1)-\Pi_2(p_1,p_2,\epsilon_2)\right]}{\partial p_1}=m-2p_1+\epsilon_1, \tag{5.3}$$

which is decreasing in p_1. Thus, $\Pi_1(p_1,p_2,\epsilon_1)-\Pi_2(p_1,p_2,\epsilon_2)$ is concave in p_1. Note that the function $f(x)=-x^-$ is increasing and concave, implying that $-\left[\Pi_1(p_1,p_2,\epsilon_1)-\Pi_2(p_1,p_2,\epsilon_2)\right]^-$ is concave. Also, $\Pi_1(p_1,p_2,\epsilon_1)$ is concave in p_1 and $\Pi_2(p_1,p_2,\epsilon_2)=p_2\left(m-p_2+\gamma p_1+\epsilon_2\right)$ is linear in p_1. Then, every term in (5.2) is concave in p_1, which assures the concavity of $S_1(p_1,p_2,\boldsymbol{\epsilon})$ (and thus $U_1(p_1,p_2,\boldsymbol{\epsilon})$) in p_1 for a given p_2. □

As an immediate result, the ex ante expected utility function $u_i(p_i,p_{-i})$, $i=1,2$, is concave in p_i as well. By Debreu (1952), we can guarantee the equilibrium existence and can solve for the equilibrium from the set of first-order conditions.

Corollary 5.1 *The price competition game in the presence of social comparison and market variability has a unique equilibrium:* $p_1=p_2=p^*\equiv\bar{p}\cap[0,p^{\max}]$, *where*

$$\bar{p}\equiv\frac{(2+\ell+e)m+2(\ell-e)\sigma}{2(2+\ell+e-\gamma)},\qquad \sigma\equiv E[\epsilon_1 1_{\{\epsilon_1<\epsilon_2\}}]\le 0, \tag{5.4}$$

$\bar{p}\cap[0,p^{\max}]$ *represents* $(\min\{\bar{p},p^{\max}\})^+$ *and* $1_{\{\cdot\}}$ *is the indicator function.*

Proof of Corollary 5.5 Please see the appendix (Section 5.7), where all proofs missing from the main-body can be found. □

Clearly, under the symmetric equilibrium, which is shown to be the *unique* equilibrium, the two agents have the same expected profit and expected utility, which will be referred to as $\pi(p^*,p^*)$ and $u(p^*,p^*)$ with the subscript i suppressed.

The term σ in (5.4) will be used for measuring market variability in place of more standard variability measures (e.g., variance) so that the price equilibrium can depend on variability in closed form. We will show that the higher the variability, the lower the value of σ. We vary market variability within a certain class of random shocks over the range $[-\bar{\alpha},\bar{\alpha}]$.

Definition 5.1 (Partial Order of Randomness) *We consider a partially ordered set* $(\mathcal{M},\preceq)$, *where* $\mathcal{M}$ *is a class of symmetrically distributed random variables on* $[L,U]$ *such that for any* $X,Y\in\mathcal{M}$, $X\preceq Y$ *(i.e., we say,* Y *is more variable than* X*) if and only if* $E(X)=E(Y)=\mu$ *and*

$$F_X(x)\begin{cases}\le F_Y(x) & \text{for } x\in[L,\mu],\\ \ge F_Y(x) & \text{for } x\in[\mu,U],\end{cases} \tag{5.5}$$

where $F_X(x)$ *and* $F_Y(x)$ *are the cumulative distribution functions of* X *and* Y *respectively.*

Note that the partial order defined as (5.5) is a sufficient condition for Y to be more variable than X in the sense of the *convex order* (for the latter, see Shaked and Shanthikumar 2007). Although the convex order does not necessarily imply the condition in (5.5), it is equivalent to Definition 5.1 for many common bounded, symmetric, unimodal distributions, such as truncated normal and uniform distributions. For example, a class of truncated normal distributions with the same mean but different standard deviations satisfies Definition 5.1, with the partial order referring to the convex order. For another example, with the same range $[-1,1]$ and the same mean 0, a two-point distribution taking values of 1 and -1 with an equal probability is more variable than a uniform distribution over $[-1,1]$. For ease of analysis, we focus on the partial order of Definition 5.1 rather than other more general stochastic orders.

The following lemma confirms that the term σ is decreasing with respect to the variability of ϵ, where ϵ is the generic random variable that represents the probability distribution of ϵ_1 and ϵ_2.

Lemma 5.2 *Given any $X, Y \in \mathcal{M}$ where $\mathcal{M}$ is a class of distributions endowed with a partial order $\preceq$ as defined in Definition 5.1. Suppose that X_i, Y_i, $i = 1, 2$, are i.i.d. copies of X and Y, respectively. Then $X \preceq Y$ if and only if $E\left\{Y_1 1_{\{Y_1<Y_2\}}\right\} \leq E\left\{X_1 1_{\{X_1<X_2\}}\right\}$.*

Proof of Lemma 5.2. Let $X, Y \in \mathcal{M}$ be supported on $[L, U]$. Let $\tilde{X} = X - (L+U)/2$ and $\tilde{Y} = Y - (L+U)/2$. It is equivalent to prove that $\tilde{X} \preceq \tilde{Y}$ if and only if $E\left\{\tilde{X}_1 1_{\{\tilde{X}_1<\tilde{X}_2\}}\right\} \leq E\left\{\tilde{Y}_1 1_{\{\tilde{Y}_1<\tilde{Y}_2\}}\right\}$, where $\tilde{X}_i$ and $\tilde{Y}_i$ $(i = 1, 2)$ are independent copies of X and Y, respectively. Without loss of generality, we can assume that $(L+U)/2 = 0$, so that $\tilde{X} = X$, $\tilde{Y} = Y$. Let $\bar{x} = (U-L)/2$ for ease of notation. Then, both X and Y are supported on $[-\bar{x}, \bar{x}]$.

First, note that $E\left\{X_1 1_{\{X_1<X_2\}}\right\} = \int_{-\bar{x}}^{\bar{x}} x\bar{F}_X(x) f_X(x)dx = \frac{1}{2}\left(-\bar{x} + \int_{-\bar{x}}^{\bar{x}} \bar{F}_X(x)^2 dx\right)$, where the last equality is due to integration by parts.

If $X \preceq Y$, by Definition 5.1, there exists a nonnegative function $\Delta(x)$ such that $\bar{F}_Y(x) = \bar{F}_X(x) - \Delta(x)$ for $x \in [-\bar{x}, 0]$, $\bar{F}_Y(x) = \bar{F}_X(x) + \Delta(x)$ for $x \in [0, \bar{x}]$, and $\Delta(x) = \Delta(-x)$ for $x \in [-\bar{x}, 0]$ (i.e., $\Delta(x)$ is symmetric on $[-\bar{x}, \bar{x}]$).

$$
\begin{aligned}
E\left\{X_1 1_{\{X_1<X_2\}}\right\} =& \frac{1}{2}\left[-\bar{x} + \int_{-\bar{x}}^{\bar{x}} \bar{F}_X(x)^2 dx\right] \\
=& \frac{1}{2}\left\{-\bar{x} + \int_{-\bar{x}}^{0} \left[\bar{F}_Y(x) + \Delta(x)\right]^2 dx + \int_{0}^{\bar{x}} \left[\bar{F}_Y(x) - \Delta(x)\right]^2 dx\right\} \\
=& \frac{1}{2}\left\{-\bar{x} + \int_{-\bar{x}}^{\bar{x}} \bar{F}_Y(x)^2 dx + \int_{-\bar{x}}^{\bar{x}} \Delta(x)^2 dx + 2\int_{-\bar{x}}^{0} \bar{F}_Y(x)\Delta(x)dx - 2\int_{0}^{\bar{x}} \bar{F}_Y(x)\Delta(x)dx\right\} \\
=& \frac{1}{2}\left\{-\bar{x} + \int_{-\bar{x}}^{\bar{x}} \bar{F}_Y(x)^2 dx + \int_{-\bar{x}}^{\bar{x}} \Delta(x)^2 dx + 2\int_{-\bar{x}}^{0} \bar{F}_Y(x)\Delta(x)dx - 2\int_{-\bar{x}}^{0} \bar{F}_Y(-x)\Delta(x)dx\right\} \\
\geq& \frac{1}{2}\left\{-\bar{x} + \int_{-\bar{x}}^{\bar{x}} \bar{F}_Y(x)^2 dx\right\} = E\left\{Y_1 1_{\{Y_1<Y_2\}}\right\},
\end{aligned}
$$

where the inequality follows from the facts that $\int_{-\bar{x}}^{\bar{x}} \Delta(x)^2 dx \geq 0$ and $\bar{F}_X(x) \geq \bar{F}_X(-x)$ for all $x \in [-\bar{x}, 0]$.

If $Y \preceq X$ and Y does not equal X in distribution (i.e., $X \preceq Y$ does not hold), then by exchanging the roles of X and Y in the above analysis we have

$$E\left\{Y_1 1_{\{Y_1<Y_2\}}\right\} = \frac{1}{2}\left\{-\bar{x} + \int_{-\bar{x}}^{\bar{x}} \bar{F}_X(x)^2 dx + \int_{-\bar{x}}^{\bar{x}} \Gamma(x)^2 dx + 2\int_{-\bar{x}}^{0} \bar{F}_X(x)\Gamma(x)dx - 2\int_{-\bar{x}}^{0} \bar{F}_X(-x)\Gamma(x)dx\right\},$$

where $\Gamma(x) = \bar{F}_Y(x) - \bar{F}_X(x)$ for $x \in [-\bar{x}, 0]$ and $\Gamma(x) = \bar{F}_X(x) - \bar{F}_Y(x)$ for $x \in [0, \bar{x}]$. Because Y does not equal X in distribution, $\Gamma(x)$ must be strictly positive over a set with positive measure. Thus, $E\left\{Y_1 1_{\{Y_1<Y_2\}}\right\} > \frac{1}{2}\left\{-\bar{x} + \int_{-\bar{x}}^{\bar{x}} \bar{F}_X(x)^2 dx\right\} = E\left\{X_1 1_{\{X_1<X_2\}}\right\}$.

Hence we have shown that $X \preceq Y$ if and only if $E\{X_1 1_{\{X_1<X_2\}}\} \geq E\{Y_1 1_{\{Y_1<Y_2\}}\}$. □

By Lemma 5.2, $\sigma \equiv E[\epsilon_1 1_{\{\epsilon_1<\epsilon_2\}}] \leq 0$ can indeed be considered as a measure of the variability of market uncertainty ϵ. The more variable the market shock, the more negative the value of σ. The measure reaches its minimum, $-\frac{\bar{\alpha}}{4}$, when ϵ follows the two-point distribution with $\Pr(\epsilon = -\bar{\alpha}) = \Pr(\epsilon = \bar{\alpha}) = \frac{1}{2}$.

5.4 Model Analysis and Results

In this section, we analyze the model in details and study the comparative statics of equilibrium behavior with respect to the extent of social comparisons and market variability. The resulting insights illustrate how demand uncertainty interacts with social comparisons in determining the competition outcomes, in contrast to the deterministic demand case. In addition, we also relax the assumption that agents' information about market variability is unbiased and examine how biased belief of market variability may affect agents' pricing and profits.

5.4.1 Effects of Social Comparison

Deterministic Demand.

As a benchmark, we consider the special case when there is no demand uncertainty.

Lemma 5.3 *If $\epsilon_1 = \epsilon_2 \equiv 0$, the symmetric equilibrium price and equilibrium profit are decreasing in the ahead-seeking parameter e and the behind-averse parameter ℓ.*[4]

Proof of Lemma 5.3. The objective function of agent 1 is

$$\begin{aligned}
u_1(p_1,p_2) =& \pi_1(p_1,p_2) + e\left[\pi_1(p_1,p_2) - \pi_2(p_1,p_2)\right]^+ - \ell\left[\pi_1(p_1,p_2) - \pi_2(p_1,p_2)\right]^- \\
=& \pi_2(p_1,p_2) + (1+e)\left[\pi_1(p_1,p_2) - \pi_2(p_1,p_2)\right] - (\ell - e)\left[\pi_1(p_1,p_2) - \pi_2(p_1,p_2)\right]^- \\
=& p_2(m - p_2 + \gamma p_1) + (1+e)(p_1 - p_2)(m - p_1 - p_2) - (\ell - e)\left[(p_1 - p_2)(m - p_1 - p_2)\right]^- .
\end{aligned}$$

[4] Throughout the chapter, monotonicity is used in its weaker sense.

If $p_2 \le \frac{m}{2}$, then

$$u_1(p_1,p_2) = \begin{cases} p_2(m-p_2+\gamma p_1) + (1+\ell)(p_1-p_2)(m-p_1-p_2) & \text{for } p_1 \le p_2, \\ p_2(m-p_2+\gamma p_1) + (1+e)(p_1-p_2)(m-p_1-p_2) & \text{for } p_2 < p_1 \le m-p_2, \\ p_2(m-p_2+\gamma p_1) + (1+\ell)(p_1-p_2)(m-p_1-p_2) & \text{for } p_1 > m-p_2. \end{cases}$$

In turn, the best response function can be decided as follows:

$$\tilde{B}(p_2) = \begin{cases} \frac{m}{2} + \frac{\gamma p_2}{2(1+e)} & \text{for } p_2 \le \frac{(1+e)m}{\gamma+2(1+e)}, \\ m - p_2 & \text{for } \frac{(1+e)m}{\gamma+2(1+e)} < p_2 \le \frac{(1+\ell)m}{\gamma+2(1+\ell)}, \\ \frac{m}{2} + \frac{\gamma p_2}{2(1+\ell)} & \text{for } p_2 > \frac{(1+\ell)m}{\gamma+2(1+\ell)}. \end{cases}$$

Similarly, we can obtain the form of the best response function for $p_2 \ge \frac{m}{2}$.

$$\tilde{B}(p_2) = \begin{cases} \frac{m}{2} + \frac{\gamma p_2}{2(1+e)} & \text{for } p_2 \ge \frac{(1+e)m}{2(1+e)-\gamma}, \\ p_2 & \text{for } \frac{(1+\ell)m}{2(1+\ell)-\gamma} < p_2 \le \frac{(1+e)m}{2(1+e)-\gamma}, \\ \frac{m}{2} + \frac{\gamma p_2}{2(1+\ell)} & \text{for } p_2 < \frac{(1+\ell)m}{2(1+\ell)-\gamma}. \end{cases}$$

Solving $\tilde{B}(p_2) = p_2$, we get infinitely many equilibra: any $p^* \in \left[\frac{m(1+\ell)}{2(1+\ell)-\gamma}, \frac{m(1+e)}{2(1+e)-\gamma}\right]$ is an equilibrium. As e increases, the upper limit of this interval decreases. Suppose that $e_1 < e_2$. For every equilibrium price $p^{2*} \in \left[\frac{m(1+\ell)}{2(1+\ell)-\gamma}, \frac{m(1+e_2)}{2(1+e_2)-\gamma}\right]$ for the case with $e = e_2$, we can find an equilibrium price $p^{1*} \in \left[\frac{m(1+\ell)}{2(1+\ell)-\gamma}, \frac{m(1+e_1)}{2(1+e_1)-\gamma}\right]$ for the case with $e = e_1$, such that $p^{2*} \le p^{1*}$. In this sense, the equilibrium price is decreasing in e. In the same sense, it is also decreasing in ℓ.

It is easy to see that every equilibrium price p^* is lower than $\widetilde{p} \equiv \frac{m}{2(1-\gamma)}$, at which $\pi(p,p)$ achieves the maximum. Thus, $\pi(p^*,p^*)$ is decreasing in e. $\square$

For deterministic demand, the social externality is in the form of

$$s(p_1,p_2) \equiv e\left[\pi_i(p_1,p_2) - \pi_{-i}(p_1,p_2)\right]^+ - \ell\left[\pi_i(p_1,p_2) - \pi_{-i}(p_1,p_2)\right]^- . \tag{5.6}$$

Lemma 5.3 confirms the conventional wisdom and social comparison theory, which suggest that in a deterministic world, both behind aversion and ahead-seeking behavior, by exerting the social externality $s(p_1,p_2)$, lead to more competitive situations. That can be explained as follows. Suppose agents stand at the equilibrium of price competition without social comparison effects. In the presence of social comparisons, other than maximizing one's own profit, each agent gains more utility by beating the competitor. Such an ahead-seeking incentive would induce one agent to undercut the competitor by reducing price. The other agent, disliking underperformance, also follows suit by reducing price. This undercutting leads to a lower price equilibrium than the one sustained in the absence of social comparisons. We call this *social competitive effect*. This socially intensified competition leads to a profit loss for both agents, compared to the situation without social comparisons.[5]

Random Demand.

We show that demand uncertainty changes the competitive landscape in the presence of social comparisons. In particular, we investigate the effect of social comparisons on the agents' price, expected profit, and expected utility in equilibrium.

Proposition 5.1 *In equilibrium, we have:*

(i) (BEHIND AVERSION) *The price p^*, the expected profit $\pi(p^*, p^*)$, and the expected utility $u(p^*, p^*)$ are decreasing in the behind-averse parameter ℓ;*

(ii) (AHEAD SEEKING) *There exists a threshold on the market variability above which the price p^*, the expected profit $\pi(p^*, p^*)$, and the expected utility $u(p^*, p^*)$ are increasing in the ahead-seeking parameter e; otherwise, the price p^* and the expected profit $\pi(p^*, p^*)$ are decreasing in e.*[6]

Proof of Proposition 5.1. For notation convenience, let $r \equiv 1/(2+\ell+e)$. By the assumption $e \geq -\ell$, it is clear that $0 < r \leq \frac{1}{2}$.

(i) PRICE. By (5.4), it is easily verified that

$$\frac{\mathrm{d}\bar{p}}{\mathrm{d}\ell} = -\frac{\gamma m - 2(2+2e-\gamma)\sigma}{2(2+\ell+e-\gamma)} \leq 0.$$

Thus, $\bar{p}$ is decreasing in ℓ and so is $p^* = \bar{p} \cap [0, p^{\max}]$.

PROFIT. It is easy to see that $\pi(p,p)$ is concave in p and reaches its maximum at $p = \widetilde{p} \equiv \frac{m}{2(1-\gamma)}$. Since $\sigma \leq 0$ and $r \in (0, \frac{1}{2}]$, we know that $\bar{p} = \frac{m+2(\ell-e)\sigma r}{2(1-\gamma r)} \leq \widetilde{p}$. This implies that $\pi(p,p)$ is increasing in $p \leq \widetilde{p}$. Then, $\pi(p^*, p^*)$ is decreasing in ℓ, because $p^*(\leq \widetilde{p})$ is decreasing in ℓ.

UTILITY. In equilibrium, the expected utility is $u(p^*, p^*) = \pi(p^*, p^*) + 2(\ell - e)p^*\sigma$. We take derivative of $(\ell - e)\bar{p}$ with respect to ℓ:

$$\frac{\mathrm{d}[(\ell-e)\bar{p}]}{\mathrm{d}\ell} = \frac{\theta_0 + \theta_1\ell + \theta_2\ell^2}{2(2+\ell+e-\gamma)^2},$$

where $\theta_0 \equiv 4m - 2\gamma m + (4m - 8\sigma + 4\gamma\sigma)e + (m - 6\sigma)e^2$, $\theta_1 \equiv 4m - 2\gamma m + 8\sigma - 4\gamma\sigma + (2m + 4\sigma)e$, $\theta_2 \equiv m + 2\sigma$. Lemma 5.2 shows that σ is non-positive and decreasing with respect to the variability of ϵ. ϵ achieves the largest variability with $\sigma = -\frac{\bar{\alpha}}{4}$ when it follows the two-point distribution satisfying $\Pr(\epsilon = -\bar{\alpha}) = \Pr(\epsilon = \bar{\alpha}) = \frac{1}{2}$. Given that $\bar{\alpha} \leq m$ (see Assumption (R)), we have $\sigma \geq -\frac{\bar{\alpha}}{4} \geq -\frac{m}{4}$.

[5] We verify that Lemma 5.3 is robust even if the agents have asymmetric market sizes. In other words, even if $d_i(p_i, p_{-i}) = m_i - p_i + \gamma p_{-i}$, $i = 1, 2$, with $m_i \neq m_{-i}$, both upward and downward social comparisons still work in the same direction to push down the equilibrium prices.

[6] The expected utility $u(p^*, p^*)$ in general does not have any monotone property in e.

Then, $\theta_i \geq 0$ for $i = 0, 1, 2$, so that $\frac{\mathrm{d}[(\ell-e)\bar{p}]}{\mathrm{d}\ell}$ is nonnegative, implying that $(\ell - e)p^*$ is increasing in ℓ. Consequently, $2(\ell - e)p^*\sigma$ decreases in ℓ. We already showed that $\pi(p^*, p^*)$ is decreasing in ℓ. Thus, $u(p^*, p^*)$ is decreasing in ℓ.

(ii) <u>PRICE</u>. To see how p^* varies in e, we take derivative:

$$\frac{\mathrm{d}\bar{p}}{\mathrm{d}e} = -\frac{2(2 + 2\ell - \gamma)\sigma + \gamma m}{2(2 + \ell + e - \gamma)^2},$$

which is positive if and only if $\sigma < -\frac{\gamma m}{2(2+2\ell-\gamma)}$. As a result, $\bar{p}$ (thus also p^*) is increasing in e if $\sigma < -\frac{\gamma m}{2(2+2\ell-\gamma)}$, and is decreasing in e otherwise.

<u>PROFIT</u>. Similar to part (i), we can conclude that $\pi(p^*, p^*)$ is increasing in e if $\sigma < -\frac{\gamma m}{2(2+2\ell-\gamma)}$, and is decreasing in e otherwise.

<u>UTILITY</u>. Consider the case with $\sigma < -\frac{\gamma m}{2(2+2\ell-\gamma)}$. As already shown, $\pi(p^*, p^*)$ is increasing in e. We examine how $(\ell - e)\bar{p}$ varies in response to increase of e.

We have

$$\frac{\mathrm{d}[(\ell - e)\bar{p}]}{\mathrm{d}e} = -\frac{\eta_0 + \eta_1 e + \eta_2 e^2}{2(2 + \ell + e - \gamma)^2},$$

with $\eta_0 = 4m + 4m\ell + 8\sigma\ell - 4\gamma\sigma\ell - 2\gamma m + 6\sigma\ell^2 + m\ell^2$, $\eta_1 = 4m - 8\sigma - 2\gamma m + 4\gamma\sigma - 4\sigma\ell + 2m\ell$ and $\eta_2 = m - 2\sigma$.

Given that $\sigma \geq -\frac{m}{4}$, we have $\eta_i \geq 0$ for $i = 0, 1, 2$. Then, $(\ell - e)\bar{p}$ is decreasing in e. Consequently, $u(p^*, p^*) = \pi(p^*, p^*) + 2(\ell - e)p^*\sigma$ is increasing in e. $\square$

Proposition 5.1 says that ceteris paribus, the more behind-averse agents are, the more intense price competition will be. Moreover, it is intriguing that when market variability is large enough, ceteris paribus, the more ahead-seeking agents are, the less intense price competition will be. In other words, ahead seeking by agents alleviates price competition in a market with sufficiently large market variability, while behind aversion behavior of agents always intensifies price competition. To understand this somewhat counterintuitive result, we take a closer look at the expected gain-loss utility under social comparisons and see how exactly social comparisons influence competitive behavior under demand uncertainty. By Equation (5.2), we can write the expected social utility of agent i, $i = 1, 2$:

$$E[S_i(p_1, p_2, \boldsymbol{\epsilon})] = \underbrace{e\left[\pi_i(p_1, p_2) - \pi_{-i}(p_1, p_2)\right]}_{\text{expected-comparison effect: } \downarrow} \underbrace{-(\ell - e)\left\{E\left[\Pi_i(p_1, p_2, \epsilon_i) - \Pi_{-i}(p_1, p_2, \epsilon_{-i})\right]^-\right\}}_{\text{variable-inequality effect: } \downarrow \text{ or } \uparrow}, \quad (5.7)$$

where $\pi_i(p_1, p_2) = p_i d_i(p_1, p_2)$ is the deterministic profit function for agent i.

The social comparisons under market uncertainty can be divided into two parts. The first externality, corresponding to the first term in the expression of $E[S_i(p_1, p_2, \boldsymbol{\epsilon})]$, captures the expected social comparison effect independently of the distribution of demand uncertainty. We call this the *expected-comparison* *effect*. By applying Lemma 5.3 to the case $e = \ell$, we can see that the expected-comparison effect intensifies price competition, as in the deterministic case. In other words, as the agentsbecome more behind-averse or ahead-seeking, the expected-comparison effect pushes the equilibrium prices lower. Since the behind aversion parameter ℓ is assumed to dominate the ahead-seeking parameter e, this expected-comparison effect is different from, and intensifies price competition to a *less* extent than, the social comparison externality $s(p_1, p_2)$ in Equation (5.6) for deterministic demand.

Moreover, the second externality, corresponding to the second term in the expression of $E[S_i(p_1, p_2, \boldsymbol{\epsilon})]$, captures the additional externality driven by market variability. We call this the *variable-inequality effect*. This effect shows how variability moderates competitive decision making. Specifically, upward and downward social comparisons would shift the weights of different uncertain scenarios in an agent's decision making under uncertainty, in a different way. In setting the ex ante prices, agents would imagine all possible outcomes of market realizations. On the one hand, the agent with a low market realization would prefer to post a relatively low price, in order to stimulate demand. On the other hand, the agent with a high market realization would prefer to post a relatively high price, in order to capitalize on the large market size. Downward and upward social comparisons play different roles in the variable-inequality effect. When ahead seeking becomes more prominent, in deciding the price ex ante, the agents put more weight on those high market realizations, because they tend to beat the competitor and gain more pleasure in those situations. Since a larger market warrants a higher price, the greater weight on those more booming market realizations would raise the ex ante prices higher. On the other hand, when behind aversion becomes more significant, the agents put more weight on those low market realizations, because they tend to feel more pain in those situations. Since a smaller market demands a lower price, the greater weight on those more depressing market realizations would push down the ex ante prices.

Combining the expected-comparison and variable-inequality effects, ceteris paribus, as behind aversion behavior becomes more prominent, both effects lead to more intense price competition. Ceteris paribus, as ahead seeking becomes more prominent, the expected-comparison effect is more pro-competitive and the variable-inequality effect is more anti-competitive. When the demand variability becomes higher, the variable-inequality effects become more prominent. If market variability is large enough, the variable-inequality effect is dominant; as a combined result, ceteris paribus, the market competition becomes milder when the agents exhibit stronger ahead seeking behavior.

Comparison Between Situations with and without Social Comparisons.

We have shown how the equilibrium performance changes in relation to the risk-averse and ahead-seeking parameters separately. In the extreme, if we set $\ell = e = 0$ in Equation (5.4), the situation with demand uncertainty is equivalent to the setting when there is no social comparison and demand uncertainty at all, namely, the traditional setting of price competition in a deterministic world (as long as the additive variability has its mean equal to zero). It is not clear from Proposition 5.1 whether the marketplace becomes more or less competitive when agents who did not practice social comparison start to receive social incentives for comparing with one another. By Proposition 5.1, when agents practice behind aversion, price competition becomes more intense. However, again by the same proposition, when agents practice ahead seeking, price competition can become less intense. It is not yet clear so far which force would dominate.

It is easy to see that, if social comparison is absent, then $p_1 = p_2 = \hat{p} \equiv \min\{p^{\max}, m/(2-\gamma)\}$ is the unique price equilibrium. In the following proposition, we compare the equilibrium price, profit, and utility with and without social comparison.

Proposition 5.2 *The equilibrium price, profit, and utility are lower with social comparison than without.*

Proof of Proposition 5.2. PRICE. We first compare $\bar{p}$ with $\frac{m}{2-\gamma}$ by calculating their difference.

$$\bar{p} - \frac{m}{2-\gamma} = \frac{m + 2(\ell - e)r\sigma}{2(1-\gamma r)} - \frac{m}{2-\gamma} = \frac{-\gamma(1-2r)m + 2(2-\gamma)(\ell-e)r\sigma}{2(1-\gamma r)(2-\gamma)} \leq 0,$$

where $r \equiv \frac{1}{2+\ell+e}$ and the inequality follows from the facts that $r \leq \frac{1}{2}$ and $\sigma \leq 0$. It follows that $p^* = \bar{p} \bigcap [0, p^{\max}] \leq \min\left\{p^{\max}, \frac{m}{2-\gamma}\right\} = \hat{p}$.

PROFIT. Because $\frac{m}{2-\gamma} \leq \frac{m}{2(1-\gamma)} = \widetilde{p}$, we have $p^* \leq \hat{p} \leq \widetilde{p}$. The function $\pi(p,p)$ is increasing in p for $p \leq \widetilde{p}$. Thus, $\pi(p^*, p^*) \leq \pi(\hat{p}, \hat{p})$.

UTILITY. In equilibrium, we have $u(p^*, p^*) = \pi(p^*, p^*) + 2(\ell - e)\sigma p^* \leq \pi(p^*, p^*) \leq \pi(\hat{p}, \hat{p})$. When social comparison is absent, we have $u(\hat{p}, \hat{p}) = \pi(\hat{p}, \hat{p})$. Then, $u(p^*, p^*) \leq u(\hat{p}, \hat{p})$. □

Proposition 5.2 shows that agents experience more intense price competition with social comparison than without. Moreover, the agents earn lower profit and utility with social comparison than without. That is because unlike the situation without social comparison, when there is social comparison the potentially anti-competitive ahead seeking is always dominated by pro-competitive behind aversion behavior, under the assumption that the behind aversion parameter ℓ dominates the ahead-seeking parameter e. As mentioned in Section 5.1, social comparison among competitors may be inevitable, though they could be better off without it.

5.4.2 Effect of Market Variability

We investigate the effect of market variability on the agents' equilibrium price, expected profit, and expected utility with social comparisons. We summarize the results in the following proposition.

Proposition 5.3 *In equilibrium, the price p^* and the expected profit $\pi(p^*, p^*)$ are decreasing in the market variability, and the expected utility $u(p^*, p^*)$ is decreasing in the market variability as long as $u(p^*, p^*)$ remains nonnegative.*

Proof of Proposition 5.3. PRICE. By (5.4), $\bar{p}$ is increasing in σ. Then, $p^* = \bar{p} \bigcap [0, p^{\max}]$ is decreasing with respect to the variability of ϵ be lemma 5.2.

<u>Profit</u>. Since $\pi(p,p)$ is concave in p and achieves the maximum at $p = \widetilde{p} \equiv \frac{m}{2(1-\gamma)}$, we know that $\pi(p,p)$ is increasing in p for $p \le \widetilde{p}$. It is easy to show that $p^* \le \widetilde{p}$. Then, $\pi(p^*,p^*)$ is increasing in p^*, thus decreasing with respect to the variability of ϵ by (i).

<u>Utility</u>. From (5.2), we can obtain the form of $u(\bar{p},\bar{p})$ as follows:

$$\begin{aligned} u(\bar{p},\bar{p}) =& \bar{p}\left[m-(1-\gamma)\bar{p}-(\ell-e)E(\epsilon_1-\epsilon_2)^-\right] = \bar{p}\left[m-(1-\gamma)\cdot\frac{m+2(\ell-e)r\sigma}{2(1-\gamma r)}+2(\ell-e)\sigma\right] \\ =& \bar{p}\cdot\frac{[1+\gamma(1-2r)]m+2(\ell-e)[2-r(1+\gamma)]\sigma}{2(1-\gamma r)}. \end{aligned}$$

Note that $r \equiv \frac{1}{2+\ell+e} \in (0,\frac{1}{2}]$ and $\gamma \in [0,1)$. Since both $\bar{p}$ and σ are decreasing with respect to the variability of ϵ, $u(\bar{p},\bar{p})$ is decreasing in the variability of ϵ as long as it remains non-negative. □

Proposition 5.3 shows that, ceteris paribus, price competition is more intense when there is more market variability. The variable-inequality effect is affected by market variability, whereas the expected-comparison effect is not. When market variability increases, both the anti-competitive ahead seeking and the pro-competitive behind aversion behavior increase. As we have shown, more prominent ahead seeking under high market variability reduces price competition, while more prominent behind aversion promotes price competition. The former is dominated by the latter, which leads to more intense price competition. This can be further explained as follows. In equilibrium, the expected utility of a agent can be written as $u(p^*,p^*) = p^*d_i(p^*,p^*) - (\ell-e)p^*E[(\epsilon_1-\epsilon_2)^-]$. When the variability of ϵ increases, the term $E[(\epsilon_1-\epsilon_2)^-] = -2\sigma \ge 0$ increases as a result of the greater possibility of having more unequal realizations of markets. It would appear logical that in the presence of social comparisons, the agents tend to lower the price p^* to mitigate the negative effects of market variability (since $\ell - e \ge 0$).

Moreover, a more intense price competition resulting from more uncertain market conditions further reduces both agents' profits. In fact, the equilibrium price p^* in the presence of social comparisons is already below the "socially optimal" price $\widetilde{p}$ (which maximizes $2\pi(p,p)$ in terms of p), i.e., the price that a centralized profit-maximizing planner would optimally set for both agents. Therefore, when p^* falls in response to greater market variability, the profit of the agents is always reduced.

Proposition 5.3 also suggests that the agents' utility diminishes as market variability increases. While the agents lower their equilibrium prices p^* in response to more uncertain market conditions, the expected social utility $s(p^*,p^*) = 2(\ell-e)\sigma p^*$ also declines (i.e., becomes more negative) with respect to market variability (since the higher the market variability, the more negative the term σ) as long as the expected utility $u(p^*,p^*)$ remains nonnegative. With the decrease of both expected profit and expected social utility, the overall expected utility $u(p^*,p^*)$ also decreases. It is possible that $u(p^*,p^*)$ will become negative when market variability reaches a certain threshold and remain negative thereafter. Since a negative utility $u(p^*,p^*)$ offers no incentive for the agents to remain in the market, the effect of market variability on $u(p^*,p^*)$ beyond that threshold may no longer be of interest to us.

5.4.3 Biased Belief about Market Variability

Intuitively, the effect of social comparisons hinges on the observability of the factors that are being compared. So far, we have assumed that each agent has the same knowledge of its own market variability and its competitor's; i.e., the distributions of ϵ_1 and ϵ_2 are identical and are common knowledge to both agents. In this extension, we consider the model in which the agents have perceived belief about market variability that may underestimate or overestimate each other's market variability; i.e., agents' perceived belief about market variability may be biased.

To this end, we use the following notation. Either of the two agents knows its own market uncertainty as ϵ and perceives the other agent's to be $\tilde{\epsilon}$. In other words, ϵ captures the "true" market uncertainty of a agent on the basis of its own forecast, while $\tilde{\epsilon}$ represents the possibly distorted market uncertainty of the competitor perceived by the agent. These two random variables need not be identically distributed. The agents overestimate each other's market variability if $\tilde{\epsilon}$ is more variable than ϵ, and underestimate it if $\tilde{\epsilon}$ is less variable than ϵ, with the comparison of variability in the sense of Definition 5.1.

It is easily verified that the agents' expected utility function is concave, thus guaranteeing the existence of equilibrium prices. In particular, the equilibrium price for both agents is given by $p^* = \bar{p} \cap [0, p^{\max}]$, where

$$\bar{p} \equiv \frac{(2+\ell+e)m + 2(\ell-e)\tilde{\sigma}}{2(2+\ell+e-\gamma)}, \tag{5.8}$$

and $\tilde{\sigma} \equiv E[\epsilon 1_{\{\epsilon<\tilde{\epsilon}\}}]$. The closed-form expression (5.8) reduces to Equation (5.4) when there are no biased beliefs. Moreover, by an analysis similar to the proof of Corollary 5.1, we can show that the equilibrium given above is the *unique* equilibrium if $e > -1$. Lemma 5.4 below shows the monotonicity of the term $\tilde{\sigma}$ and its counterpart $\breve{\sigma} \equiv E[\tilde{\epsilon} 1_{\{\tilde{\epsilon}<\epsilon\}}]$ in the market distributions ϵ and $\tilde{\epsilon}$.

Lemma 5.4 *The terms $\tilde{\sigma}$ and $\breve{\sigma}$ satisfy the following monotone properties:*

(i) *$\tilde{\sigma}$ is nonpositive, decreasing in the variability of ϵ, and increasing in the variability of $\tilde{\epsilon}$;*

(ii) *If the density functions of ϵ and $\tilde{\epsilon}$, namely $f_\epsilon(x)$ and $f_{\tilde{\epsilon}}(x)$, are unimodal on $[-\bar{\alpha}, \bar{\alpha}]$, then $2\tilde{\sigma} + \breve{\sigma}$ is decreasing in the variability of $\tilde{\epsilon}$ and $\tilde{\sigma} + 2\breve{\sigma}$ is decreasing in the variability of ϵ.*

Proof of Lemma 5.4. (i) We write $E\left[\epsilon 1_{\{\epsilon<\tilde{\epsilon}\}}\right] = \int_{-\bar{\alpha}}^{\bar{\alpha}} x \bar{F}_{\tilde{\epsilon}}(x) f_\epsilon(x) dx$, where $\bar{F}_{\tilde{\epsilon}}(x) = 1 - F_{\tilde{\epsilon}}(x) \equiv \Pr(\tilde{\epsilon} > x)$ and $f_\epsilon(x)$ is the density function of ϵ. To prove that $E\left[\epsilon 1_{\{\epsilon<\tilde{\epsilon}\}}\right]$ is increasing in the variability of $\tilde{\epsilon}$, we consider a random variable $\tilde{\epsilon}'$, such that $\tilde{\epsilon} \preceq \tilde{\epsilon}'$. That is, there exists a symmetric function $\Delta(x) \geq 0$ defined on $[-\bar{\alpha}, \bar{\alpha}]$ such that $F_{\tilde{\epsilon}'}(x) = F_{\tilde{\epsilon}}(x) + \Delta(x)$ for $x \in [-\bar{\alpha}, 0]$ and $F_{\tilde{\epsilon}'}(x) = F_{\tilde{\epsilon}}(x) - \Delta(x)$ for $x \in [0, \bar{\alpha}]$. Then,

$$E\left[\epsilon 1_{\{\epsilon<\tilde{\epsilon}'\}}\right] = \int_{-\bar{\alpha}}^{0} x\left[\bar{F}_{\tilde{\epsilon}}(x) - \Delta(x) f_\epsilon(x)\right] dx + \int_{0}^{\bar{\alpha}} x\left[\bar{F}_{\tilde{\epsilon}}(x) + \Delta(x) f_\epsilon(x)\right] dx$$

$$
\begin{aligned}
&= \int_{-\bar{\alpha}}^{\bar{\alpha}} x\bar{F}_{\tilde{\epsilon}}(x)f_{\epsilon}(x)dx - \int_{-\bar{\alpha}}^{0} x\Delta(x)f_{\epsilon}(x)dx + \int_{0}^{\bar{\alpha}} x\Delta(x)f_{\epsilon}(x)dx \\
&\geq \int_{-\bar{\alpha}}^{\bar{\alpha}} x\bar{F}_{\tilde{\epsilon}}(x)f_{\epsilon}(x)dx = E\left[\epsilon 1_{\{\epsilon<\tilde{\epsilon}\}}\right].
\end{aligned}
$$

Therefore, $E\left[\epsilon 1_{\{\epsilon<\tilde{\epsilon}\}}\right]$ is increasing in the variability of $\tilde{\epsilon}$.

We continue to prove that $E\left[\epsilon 1_{\{\epsilon<\tilde{\epsilon}\}}\right]$ is decreasing in the variability of ϵ. Using integration by parts, we can show that $E\left[\epsilon 1_{\{\epsilon<\tilde{\epsilon}\}}\right] = \int_{-\bar{\alpha}}^{\bar{\alpha}} F_{\epsilon}(x)h(x)dx$, where $h(x) = xf_{\tilde{\epsilon}}(x) - \bar{F}_{\tilde{\epsilon}}(x)$. It is easy to see that $h(-x) \leq h(x)$ for all $x \geq 0$. Then, for a random variable ϵ' such that $\epsilon \preceq \epsilon'$, we have $F_{\epsilon'}(x) = F_{\epsilon}(x)+\Lambda(x)$ for $x \in [-\bar{\alpha}, 0]$ and $F_{\epsilon'}(x) = F_{\epsilon}(x) - \Lambda(x)$ for $x \in [0, \bar{\alpha}]$, where $\Lambda(x) \geq 0$ is a symmetric function defined on $[-\bar{\alpha}, \bar{\alpha}]$. Consequently,

$$
\begin{aligned}
E\left[\epsilon 1_{\{\epsilon'<\tilde{\epsilon}\}}\right] &= \int_{-\bar{\alpha}}^{0} [F_{\epsilon}(x) + \Lambda(x)]h(x)dx + \int_{0}^{\bar{\alpha}} [F_{\epsilon}(x) - \Lambda(x)]h(x)dx \\
&= \int_{-\bar{\alpha}}^{\bar{\alpha}} F_{\epsilon}(x)h(x)dx + \int_{-\bar{\alpha}}^{0} \Lambda(x)h(x)dx - \int_{0}^{\bar{\alpha}} \Lambda(x)h(x)dx \\
&= E\left[\epsilon 1_{\{\epsilon<\tilde{\epsilon}\}}\right] + \int_{0}^{\bar{\alpha}} \Lambda(x)h(-x)dx - \int_{0}^{\bar{\alpha}} \Lambda(x)h(x)dx \leq E\left[\epsilon 1_{\{\epsilon<\tilde{\epsilon}\}}\right].
\end{aligned}
$$

Thus, $E\left[\epsilon 1_{\{\epsilon<\tilde{\epsilon}\}}\right]$ is decreasing in the variability of ϵ.

Finally, since $\tilde{\sigma} = E\left[\epsilon 1_{\{\epsilon<\tilde{\epsilon}\}}\right] = 0$ if $\epsilon \equiv 0$, we have $\tilde{\sigma} \leq 0$ if ϵ is random.

(ii) We write $2\tilde{\sigma} + \check{\sigma}$ as follows:

$$
\begin{aligned}
2\tilde{\sigma} + \check{\sigma} &= 2\int_{-\bar{\alpha}}^{\bar{\alpha}} x\bar{F}_{\tilde{\epsilon}}(x)f_{\epsilon}(x)dx + \int_{-\bar{\alpha}}^{\bar{\alpha}} x\bar{F}_{\epsilon}(x)f_{\tilde{\epsilon}}(x)dx \\
&= -2\int_{-\bar{\alpha}}^{\bar{\alpha}} xF_{\tilde{\epsilon}}(x)f_{\epsilon}(x)dx + \left[-\int_{-\bar{\alpha}}^{\bar{\alpha}} F_{\tilde{\epsilon}}(x)\bar{F}_{\epsilon}(x)dx + \int_{-\bar{\alpha}}^{\bar{\alpha}} xf_{\epsilon}(x)F_{\tilde{\epsilon}}(x)dx\right] \\
&= -\int_{-\bar{\alpha}}^{\bar{\alpha}} F_{\tilde{\epsilon}}(x)\left[xf_{\epsilon}(x) + \bar{F}_{\epsilon}(x)\right]dx.
\end{aligned}
$$

Let $g(x) = xf_{\epsilon}(x) + \bar{F}_{\epsilon}(x)$. Then, $g'(x) = xf'_{\epsilon}(x)$, which is non-positive for all $x \in [-\bar{\alpha}, \bar{\alpha}]$, due to the assumption that $f_{\epsilon}(x)$ is symmetric and unimodal on $[-\bar{\alpha}, \bar{\alpha}]$. As a result, $g(x)$ is decreasing on $[-\bar{\alpha}, \bar{\alpha}]$. Following the same analysis we used in (i) for proving $\tilde{\sigma}$ is decreasing in the variability of ϵ, we can show that $2\tilde{\sigma} + \check{\sigma}$ is decreasing in the variability of $\tilde{\epsilon}$.

Symmetrically, it follows that $\tilde{\sigma} + 2\check{\sigma}$ is decreasing in the variability of ϵ. □

The following result shows how the agent's own market variability and the perceived variability about the competitor affect the equilibrium price and expected profit.

Proposition 5.4 *When agents have biased belief about market variability, in equilibrium, the price p^* and the expected profit $\pi(p^*, p^*)$ are decreasing in the agent's own market variability, and are increasing in the perceived market variability of the competitor.*

Proof of Proposition 5.4. (i) <u>Price</u>. By (5.8), p^* is increasing in $\tilde{\sigma}$. Thus, it is decreasing in the variability of ϵ and increasing in the variability of $\tilde{\epsilon}$ by Lemma 5.4.

<u>PROFIT</u>. It is easy to see that $p^* \leq \widetilde{p} \equiv \frac{m}{2(1-\gamma)}$. Since $\pi(p,p)$ is concave and reaches the maximum at $p = \widetilde{p}$, the expected profit $\pi(p^*, p^*)$ is increasing in p^*. Therefore, it is decreasing in the variability of ϵ and increasing in the variability of $\tilde{\epsilon}$.

(ii) If both agents set their prices to p, the expected social utility of either agent is $s(p,p) = eE\left[\Pi(p,p,\epsilon) - \Pi(p,p,\tilde{\epsilon})\right]^+ + \ell E\left[\Pi(p,p,\epsilon) - \Pi(p,p,\tilde{\epsilon})\right]^- = -(\ell - e)pE(\epsilon - \tilde{\epsilon})^- = (\ell - e)p(\tilde{\sigma} + \breve{\sigma})$. Then, the expected utility of the agent is $u(p,p) = \pi(p,p) + s(p,p) = p\left[m - (1-\gamma)p + (\ell - e)(\tilde{\sigma} + \breve{\sigma})\right]$. When $p = \bar{p}$,

$$\begin{aligned} u(\bar{p},\bar{p}) = \bar{p}\left[m - (1-\gamma)\bar{p} - (\ell - e)(\tilde{\sigma} + \breve{\sigma})\right] =& \bar{p}\left[m - (1-\gamma)\frac{m + 2(\ell - e)r\tilde{\sigma}}{2(1-\gamma r)} + (\ell - e)(\tilde{\sigma} + \breve{\sigma})\right] \\ =& \bar{p}\left[\frac{1 + \gamma(1-2r)}{2(1-\gamma r)}a + (\ell - e)\left(\frac{1-r}{1-\gamma r}\tilde{\sigma} + \breve{\sigma}\right)\right], \end{aligned}$$

where $r \equiv \frac{1}{2+\ell+e} \in (0, \frac{1}{2}]$. By (i), $\bar{p}$ is decreasing in the variability of ϵ. On the other hand, it is easy to see that $\frac{1-r}{1-\gamma r} > \frac{1}{2}$. Thus, $\frac{1-r}{1-\gamma r}\tilde{\sigma} + \breve{\sigma} = \left(\frac{1-r}{1-\gamma r} - \frac{1}{2}\right)\tilde{\sigma} + \frac{1}{2}(\tilde{\sigma} + 2\breve{\sigma})$, which is decreasing in the variability of ϵ by Lemma 5.4. Therefore, $u(\bar{p},\bar{p})$, thus also $u(p^*,p^*)$, is decreasing in the variability of ϵ as long as it remains nonnegative.

It remains to show that $u(p^*,p^*)$ is decreasing in the variability of $\tilde{\epsilon}$. Let us calculate the derivative of $u(\bar{p},\bar{p})$ with respect to $\tilde{\sigma}$. Using the expression $\bar{p} = \frac{m+2(\ell-e)r\tilde{\sigma}}{2(1-\gamma r)}$, we have

$$\begin{aligned} \frac{\mathrm{d}u(\bar{p},\bar{p})}{\mathrm{d}\tilde{\sigma}} =& \frac{\ell - e}{2(1-\gamma r)^2}\left\{\left[(1-\gamma r)\frac{\mathrm{d}\breve{\sigma}}{\mathrm{d}\tilde{\sigma}} + 1 + \gamma r(1-2r)\right]m \right. \\ & \left. + 2(\ell - e)r\tilde{\sigma}\left[2(1-r) + (1-\gamma r)\frac{\mathrm{d}\breve{\sigma}}{\mathrm{d}\tilde{\sigma}}\right] + 2(\ell - e)r(1-\gamma r)\breve{\sigma}\right\}. \end{aligned}$$

By Lemma 5.4, $\tilde{\sigma}$ is increasing in the variability of $\tilde{\epsilon}$ and $2\tilde{\sigma} + \breve{\sigma}$ is decreasing in the variability of $\tilde{\epsilon}$. This implies that $2\tilde{\sigma} + \breve{\sigma}$ is decreasing in $\tilde{\sigma}$. Therefore, when the variability of $\tilde{\epsilon}$ increases, $\frac{\mathrm{d}\breve{\sigma}}{\mathrm{d}\tilde{\sigma}} \leq -2$. Consequently, $(1-\gamma r)\frac{\mathrm{d}\breve{\sigma}}{\mathrm{d}\tilde{\sigma}} + 1 + \gamma r(1-2r) \leq -2(1-\gamma r) + 1 + \gamma r(1-2r) = -1 + \gamma r(3-2r) < -1 + \gamma \leq 0$, where the second inequality holds because $r(3-2r) < 1$ for $r < \frac{1}{2}$. This result implies that $\frac{\mathrm{d}u(\bar{p},\bar{p})}{\mathrm{d}\tilde{\sigma}}$ is decreasing in m. We can calculate that

$$\frac{\mathrm{d}u(\bar{p},\bar{p})}{\mathrm{d}\tilde{\sigma}}\Big|_{m=-2(\ell-e)r\tilde{\sigma}} = \frac{[(1-2r)\tilde{\sigma} + \breve{\sigma}]r(\ell - e)^2}{2(1-\gamma r)}.$$

Thus, $\frac{\mathrm{d}u(\bar{p},\bar{p})}{\mathrm{d}\tilde{\sigma}} \leq \frac{[(1-2r)\tilde{\sigma}+\breve{\sigma}]r(\ell-e)^2}{2(1-\gamma r)} \leq 0$ for $m \geq -2(\ell - e)r\tilde{\sigma}$, implying that $u(\bar{p},\bar{p})$ is decreasing in $\tilde{\sigma}$, thus in the variability of $\tilde{\epsilon}$, as long as $m \geq -2(\ell - e)r\tilde{\sigma}$ (which always holds under our assumptions, by a similar analysis as in the proof of Corollary 5.1). □

The variable-inequality effect is affected in different ways by a agent's own market variability and by what it believes to be the competitor's market variability. As is consistent with Proposition 5.3, Proposition 5.4 shows that price competition becomes more intense if the agent's own market becomes more variable. When a agent's own market is more variable, as explained before for the variable-inequality effect, the ahead seeking tends to push the prices up and the behind aversion tends to push the prices down. The former is dominated by the latter. As a combined result, the equilibrium prices are reduced and price competition is intensified.

Moreover, price competition is *alleviated* when the perceived market variability of the competitor becomes *higher*. That is because the agent would prefer to believe that the competitor experienced more extreme market shocks. On the one hand, it is more likely that the competitor has a larger realized demand; as explained before for the variable-inequality effect, the competitor's ahead seeking tends to push up the competitor's price; in anticipation of this, the agent herself has an incentive to raise its own price. On the other hand, it is more likely that the competitor has a smaller realized demand; the agent's own ahead-seeking behavior tends to push up its price. As a result, price competition is alleviated and the agent's expected profit is higher when the perceived market variability of the competitor increases.

5.5 Extensions

In this section, we consider a couple of extensions of our base model. These results not only confirm the robustness of the insights from the base model, but also enrich the understanding of the expected-comparison and variable-inequality effects that have been identified.

5.5.1 Multiplicative Demand Shock

In the base model, the demand uncertainty is in the form of an additive shock. In this subsection, we consider demand uncertainty in the form of a multiplicative shock. In particular, we consider the following demand function:

$$D_i(p_i, p_{-i}, \zeta_i) = \zeta_i d_i(p_i, p_{-i}) = \zeta_i(m - p_i + \gamma p_{-i}),$$

where ζ_1 and ζ_2 are i.i.d. and symmetrically distributed nonnegative random variables. Let ζ be the generic random variable for ζ_1 and ζ_2. We assume that $E\zeta = 1$ and $\zeta \in [2 - \bar{\beta}, \bar{\beta}]$, with $\bar{\beta} \in (1, 2]$ being the upper bound of the random shock.

As in the base model, we can define and write in closed form the ex post profit function $\Pi_i(p_1, p_2, \zeta_i)$, the expected profit function $\pi_i(p_1, p_2)$, the ex post utility function $U_i(p_1, p_2, \boldsymbol{\zeta})$, and the expected utility function $u_i(p_1, p_2)$ for the two agents $i = 1, 2$. We can show that the expected utility $u_i(p_1, p_2)$ is concave in p_i, and hence there exists a symmetric equilibrium in which

$$p_1 = p_2 = p^* = \frac{[1 + e + (\ell - e)\delta]m}{2(1 - \gamma)[1 + e + (\ell - e)\delta] + \gamma(1 + \ell + e)}, \tag{5.9}$$

where $\delta \equiv E\left[\zeta_1 1_{\{\zeta_1 < \zeta_2\}}\right]$. Because the multiplicative shock is nonnegative, $\delta \geq 0$. If we focus on a class of random shocks as in Definition 5.1 with $L = 2 - \bar{\beta}$ and $U = \bar{\beta}$, by Lemma 5.2, δ is a measure of the demand variability within the focal class of random shocks. Again, the more variable the demand shocks, the lower the value of $\delta \geq 0$. The measure reaches its maximum value $\frac{1}{2}$ when $\zeta \equiv 1$, and its minimum value $\frac{3-\bar{\beta}}{4}$ when ζ follows the two-point distribution with $\Pr(\zeta = 2 - \bar{\beta}) = \Pr(\zeta = \bar{\beta}) = \frac{1}{2}$.

The following proposition describes how the social comparison parameters ℓ and e affect the price, the expected profit, and the expected utility of the agents in equilibrium, for the alternative form of demand uncertainty.

Corollary 5.2 *In equilibrium, for substitutable products with multiplicative demand shocks:*

(i) (BEHIND AVERSION) *The price p^* and the expected profit $\pi(p^*, p^*)$ are decreasing in the behind-averse parameter ℓ, and the expected utility $u(p^*, p^*)$ is decreasing in ℓ until it reaches zero, and remains nonpositive thereafter;*

(ii) (AHEAD SEEKING) *There exists a threshold on the market variability above which the price p^* and the expected profit $\pi(p^*, p^*)$ and the expected utility $u(p^*, p^*)$ are increasing in the ahead-seeking parameter e; otherwise, the price p^* and the expected profit $\pi(p^*, p^*)$ are decreasing in e.*

Proof of Corollary 5.2. (i) <u>PRICE</u>. We calculate the derivative of p^* with respect to ℓ as follows:

$$\frac{\mathrm{d}p^*}{\mathrm{d}\ell} = \frac{-\gamma m[1 - \delta + (1 - 2\delta)e]}{\{2(1-\gamma)[1 + e + (\ell - e)\delta] + \gamma(1 + \ell + e)\}^2} < 0,$$

where the inequality holds because $0 \leq \delta \leq \frac{1}{2}$ (note that $0 \leq E(\zeta_1 - \zeta_2)^- = 1 - 2\delta$). Then, p^* is decreasing in ℓ.

<u>PROFIT</u>. We can verify that $p^* \leq \widetilde{p} \equiv \frac{m}{2(1-\gamma)}$. Then, $\pi(p^*, p^*)$ is decreasing in ℓ.

<u>UTILITY</u>. In equilibrium, the expected utility can be written into the following form:

$$u(p^*, p^*) = \pi(p^*, p^*)\,[1 - (\ell - e)(1 - 2\delta)]\,.$$

We have shown that $\pi(p^*, p^*)$ is decreasing in ℓ. Moreover, $1 - (\ell - e)(1 - 2\delta)$ is decreasing in ℓ. If $1 - (\ell - e)(1 - 2\delta) \geq 0$, then $u(p^*, p^*)$ is decreasing in ℓ.

(ii) <u>PRICE</u>. We calculate the derivative of p^* with respect to e as follows:

$$\frac{\mathrm{d}p^*}{\mathrm{d}e} = \frac{-\gamma m[(2\ell + 1)\delta - \ell]}{\{2(1-\gamma)[1 + e + (\ell - e)\delta] + \gamma(1 + \ell + e)\}^2}.$$

We see that $\frac{\mathrm{d}p^*}{\mathrm{d}e} \leq 0$ if $\delta \geq \frac{\ell}{2\ell+1}$, and $\frac{\mathrm{d}p^*}{\mathrm{d}e} > 0$ otherwise. Thus, p^* is decreasing in e if the variability of ζ is lower than the threshold at which $\delta = \frac{\ell}{2\ell+1}$, and increasing in e otherwise.

<u>PROFIT</u>. Since $p^* \leq \widetilde{p} \equiv \frac{\gamma}{2(1-\gamma)}$, $\pi(p^*, p^*)$ has the same monotonicity as p^* does in the variability of ζ.

<u>UTILITY</u>. If $\delta < \frac{\ell}{2\ell+1}$, then $u(p^*, p^*) = \pi(p^*, p^*)\,[1 - (\ell - e)(1 - 2\delta)]$ is increasing in e if it is

nonnegative. □

Corollary 5.2 confirms that impacts of social comparisons obtained under the additive demand shocks (see Proposition 5.1) are robust. In particular, more prominent behind aversion behavior always leads to more intense price competition. However, price competition is reduced by more prominent status seeking if the market variability is above a threshold, but otherwise it is intensified. These results can be explained analogously by the expected-comparison effect and the variable-inequality effect. We further proceed to examine the effects of market variability.

Corollary 5.3 *In equilibrium, for substitutable products with multiplicative demand shocks, the price p^* and the expected profit $\pi(p^*, p^*)$ are decreasing in the market variability, and the expected utility $u(p^*, p^*)$ is decreasing in the market variability before it reaches zero, and remains nonpositive thereafter.*

Proof of Corollary 5.3. It follows directly from (5.9) that p^* is increasing in δ, thus decreasing in the variability of ζ. Then, by the result that $p^* \leq \widetilde{p} \equiv \frac{m}{2(1-\gamma)}$, the expected profit $\pi(p^*, p^*)$ is decreasing in the variability of ζ.

Finally, because both $\pi(p^*, p^*)$ and $1-(\ell-e)(1-2\delta)$ are increasing in δ, the expected utility $u(p^*, p^*) = \pi(p^*, p^*)\,[1-(\ell-e)(1-2\delta)]$ increases in δ if it is nonnegative. Therefore, $u(p^*, p^*)$ is decreases in the variability of ζ until it reaches zero. □

The results in Corollary 5.3 are consistent with Proposition 5.3, showing that when the market variability increases, price competition is more intense, with lower expected profit and utility.

5.5.2 Complementary Products

Consider the model with $\gamma \in (-1, 0)$, in which the two agents sell complementary products instead of substitutable products. We now investigate how social comparisons affect the agents' equilibrium price, expected profit, and expected utility for complementary products. The nature of the game has changed, and that is expected to change how social comparisons affect equilibrium outcomes.

We can show that there exists an equilibrium with $p_1 = p_2 = p^*$, where p^* is again given by Equation (5.4). In other words, the closed form of equilibrium is intact when γ becomes negative. Moreover, it is the unique equilibrium if $e \geq -\frac{1}{2}$, following an analysis similar to the proof of Corollary 5.1.

Corollary 5.4 *In equilibrium, for complementary products, we have:*

(i) (Behind Aversion) *The price p^* is decreasing in ℓ if market variability is above a threshold, and is increasing in ℓ otherwise. The expected profit $\pi(p^*, p^*)$ is quasi-concave in ℓ (with a changeover point that possibly equals $-\infty$ or ∞);*

(ii) (Ahead Seeking) *The price p^* is increasing in e. The expected profit $\pi(p^*, p^*)$ is quasi-concave in e (with a changeover point that possibly equals $-\infty$ or ∞).*

Again, we interpret the results in Corollary 5.4 from the perspectives of the expected-comparison effect and the variable-inequality effect. We rewrite the expected social utility (5.7) in an equivalent form:

$$E[S_i(p_1,p_2,\boldsymbol{\epsilon})] = \underbrace{l\left[\pi_i(p_1,p_2)-\pi_{-i}(p_1,p_2)\right]}_{\text{expected-comparison effect: } \uparrow} \underbrace{-(\ell-e)\left\{E\left[\Pi_i(p_1,p_2,\epsilon_i)-\Pi_{-i}(p_1,p_2,\epsilon_{-i})\right]^+\right\}}_{\text{variable-inequality effect: } \uparrow \text{ or } \downarrow}.$$

The expected-comparison effect captures the externality of social comparisons imposed on the competitive price problem on expectation, independently from the distribution of demand uncertainty. This effect influences equilibrium prices for complementary products in the *opposite* direction from the case of substitutable products.

Lemma 5.5 below shows that the equilibrium prices increase when the agents' behind aversion or ahead seeking becomes more prominent under deterministic demand.

Lemma 5.5 *If $\epsilon_1 = \epsilon_2 = 0$, in equilibrium, for complementary products the price p^* increases in ℓ and e.*

Proof of Lemma 5.5. As in Lemma 5.3, we can show that any $p^* \in \left[\frac{a(1+\ell)}{2(1+\ell)-\gamma}, \frac{a(1+e)}{2(1+e)-\gamma}\right]$ is an equilibrium. Given that $\gamma < 0$, the upper limit is increasing in ℓ and the lower limit is in increasing in e. Therefore, the equilibrium price is increasing in ℓ and in e, in the sense that the likelihood of having a larger p^* is increased as ℓ and e becomes larger. □

By raising the price of a complementary product in a deterministic environment, an agent can undercut the competitor's demand and profit to reduce her feeling of loss at being outperformed by the competitor or to increase her feeling of gain by beating the competitor. However, increased equilibrium prices do not necessarily lead to a higher profit for complementary products. In fact, the equilibrium profit still decreases due to behind aversion and ahead seeking behavior, and it may increase *only if* the agents exhibit distributively fair behavior ($e < 0$). In a deterministic world, the expected-comparison effect pushes the equilibrium prices in the opposite directions for substitutable and complementary products, but the profit implication is consistent for $e, \ell \geq 0$: the agents' profit is hurt because of social comparisons.

By applying $e = \ell$, Lemma 5.5 implies that the expected-comparison effect pushes up the equilibrium prices for complementary products. In the environment with random demand, the agents are affected not only by the expected-comparison effect but also by the variable-inequality effect. The variable-inequality effect works exactly as before. When ahead seeking is more prominent, the agents put more weight on those promising demand scenarios, and that raises the ex ante prices. When behind aversion is more prominent, the agents put more weight on those depressing demand scenarios, and that suppresses the ex ante prices. As the market variability increases, both incentives become stronger. When the expected-comparison and variable-inequality effects are combined, it can be seen that, ceteris paribus, as ahead seeking becomes more prominent, both effects push up equilibrium prices. Ceteris paribus, as behind aversion becomes more prominent, the expected-comparison effect pushes up the equilibrium prices and the variable-inequality effect pushes down the equilibrium prices. When market variability is large enough, the variable-inequality effect is dominant; ceteris paribus, as a combined result, the equilibrium prices fall when the agents exhibit stronger behind aversion.

We now examine the influence of the market variability on the agents that sell complementary products, finding implications for the agents' profitability that are different from the case of substitutable products.

Proposition 5.5 *In equilibrium, for complementary products, we have:*

(i) *The price p^* is decreasing in the market variability;*

(ii) *The expected profit $\pi(p^*, p^*)$ is increasing in the market variability when it is below a threshold, and is decreasing when it is above this threshold;*

(iii) *The expected utility $u(p^*, p^*)$ is decreasing in the market variability as long as it remains nonnegative.*

Proof of Proposition 5.5. (i) and (iii) can be proved in the same way we prove Proposition 5.3. It remains to prove (ii).

It is easy to see that $p^* \geq \widetilde{p} \equiv \frac{m}{2(1-\gamma)}$ if and only if $\sigma \geq \frac{\gamma(1-r)m}{2(\ell-e)(1-\gamma)r}$. Then, $\pi(p^*, p^*)$ is decreasing in p^*, thus it is increasing in the variability of ϵ if $\sigma \geq \frac{\gamma(1-r)m}{2(\ell-e)(1-\gamma)r}$, and is decreasing in the variability of ϵ if $\sigma < \frac{\gamma(1-r)m}{2(\ell-e)(1-\gamma)r}$. $\square$

The market variability has an influence only on the variable-inequality effect. As market variability increases, in the variable-inequality effect ahead seeking pushes equilibrium prices up and behind aversion pushes equilibrium prices down. Under the assumption that the behind aversion parameter dominates the ahead-seeking parameter, as a combined result, the equilibrium price decreases in the market variability. However, when the market variability is below a certain threshold, the equilibrium price p^* in the presence of social comparisons exceeds the "socially optimal" price $\widetilde{p}$, which is the price that a centralized profit-maximizing planner would optimally set for both agents. This is because social comparison have a tendency to push up prices for complementary products (see Lemma 5.5 for the extreme case when there is no market variability). Thus, surprisingly, for market variabilities that are low enough, in the presence of social comparisons the expected profit $\pi(p^*, p^*)$ is *improved*, as a result of reduced equilibrium prices but still above $\widetilde{p}$, when the market variability *increases*. As in the case of substitutable products, larger market variability always leads to lower expected utility for the agents.

5.5.3 General Demand Curves

Finally, we extend our base model to account for non-linear demand functions. We make the following assumption on the general demand curves to guarantee the equilibrium existence.

Assumption 5.2 *(General Demand Function). The demand function $d_i(p_1, p_2)$ is twice differen-*

tiable, and

$$2\frac{\partial d_i(p_1,p_2)}{\partial p_i} + p_i\frac{\partial^2 d_i(p_1,p_2)}{\partial p_i^2} \le 0,$$
$$2\frac{\partial d_i(p_1,p_2)}{\partial p_i} + p_i\frac{\partial^2 d_i(p_1,p_2)}{\partial p_i^2} - p_{-i}\frac{\partial^2 d_{-i}(p_1,p_2)}{\partial p_i^2} \le 0,$$

for all $p_1, p_2 \in [0, p^{\max}]$*,* $i = 1, 2$*.*

Assumption 5.2 consists of the second-order conditions that ensure the concavity of both $\Pi_i(p_1, p_2, \epsilon_i)$ and $\Pi_i(p_1, p_2, \epsilon_i) - \Pi_{-i}(p_1, p_2, \epsilon_{-i})$ with respect to p_i, which further implies the concavity of $EU_i(p_1, p_2, \boldsymbol{\epsilon})$ with respect to p_i. Thus a pure Nash equilibrium exists under Assumption 5.2. The assumption will hold if either one of the following more intuitive conditions is true:

(i) The expected profit $\pi_i(p_1, p_2) = p_i d_i(p_1, p_2)$ is concave in p_i and $d_{-i}(p_1, p_2)$ is convex in p_i.

(ii) The second-order effects $\partial^2 d_i(p_1, p_2)/\partial p_i^2$ and $\partial^2 d_{-i}(p_1, p_2)/\partial p_i^2$ is much less significant compared with the first-order effect $\partial d_i(p_1, p_2)/\partial p_i$ over all $p_1, p_2 \in [0, p^{\max}]$.

The following proposition confirms the robustness of the key result in the base model, Proposition 5.1, for general demand functions.

Corollary 5.5 *Suppose Assumption 5.2 holds. In addition, suppose that* $d_i(p, p) + p\partial d_i(p, p)/\partial p_i$*,* $d_i(p, p) + p\partial d_i(p, p)/\partial p_i - p\partial d_{-i}(p, p)/\partial p_i$ *and* $d_i(p, p) + p\partial d_i(p, p)/\partial p_i + p\partial d_{-i}(p, p)/\partial p_i$ *are decreasing in* p*. Then, the following results hold in equilibrium:*

(i) The price p^* *and either agent's profit* $\pi_i(p^*, p^*)$ *are decreasing in the behind-averse parameter* ℓ*;*

(ii) The price p^* *and either agent's profit* $\pi_i(p^*, p^*)$ *are increasing in the ahead-seeking parameter* e *when the market variability is above a threshold (i.e.,* $\sigma < \hat{\sigma}$ *for some* $\hat{\sigma}$*); otherwise,* p^* *and* $\pi_i(p^*, p^*)$ *are decreasing in* e*.*

Corollary 5.5 confirms that the main findings with linear demand function still hold when demand is in a more general non-linear form. That is, consistent with traditional wisdom, the more behind-averse agents are, the more intense price competition will be. However, when market variability is large enough, the more ahead-seeking agents are, the less intense price competition will be.

5.6 Discussions and Conclusion

In this chapter, we examine how how social comparisons and demand uncertainty interact to influence competing agents' pricing strategies, and find that the two types of social comparisons, upward comparison (behind aversion) and downward comparison (ahead seeking) may have different effects on price competition as demand uncertainty changes. Consistent with the literature, upward comparison induces more intense price competition. However, different from the traditional wisdom, our research suggests that downward comparison can lead to less competition when the market variability is large enough. We identify the expected-comparison effect and the variable-inequality effect to explain the phenomenon. Although such finding is robust across several extensions such as with multiplicative demand shock or general demand function, some other factors may have certain influence on price competition as well, about which we provided brief discussions below.

Distributive Fairness. Unless otherwise specified with additional conditions, our results hold for the distributively fair behavior, namely, for the case $e < 0$, as long as $|e| \leq \ell$. In particular, those comparative statics hold with respect to the parameter $e < 0$. When $e < 0$, an increase in the parameter e means less prominent distributively fair behavior. This can be *equivalently* interpreted as more ahead seeking relative to a very prominent distributively fair benchmark.

Demand Correlation. In the base model, we assume that the demand shocks experienced by both agents are independently distributed. Suppose demand shocks are negatively correlated. When one agent experiences a large market realization, its competitor is more likely to experience a small market realization. The ahead seeking by the agent with a large market realization now gives it a stronger incentive to raise its price, as compared to independently distributed demand shocks. Similarly, the behind aversion of an agent with a small market realization now has a stronger incentive to lower its price. In other words, social comparison behavior exerts a *stronger* externality in the variable-inequality effect if market shocks are *negatively* correlated; the qualitative implications of social comparisons are expected to remain but to a larger magnitude. Analogously, social comparison behavior may exert a *weaker* externality in the variable-inequality effect if market shocks are *positively* correlated.[7]

Asymmetric Market. As mentioned, both downward and upward social comparison behavior push down the prices in a deterministic environment with asymmetric markets. As a result, in the expected-comparison effect, social comparisons push down the prices. In the variable-inequality effect, the qualitative implications of social comparisons still hold for asymmetric markets. However, there are two notable differences. First, behind aversion becomes less relevant to the larger agent, while status seeking becomes less relevant to the smaller agent. Second, social comparisons have an effect, to a smaller extent on the stochastically larger agent, and to a larger extent, on the stochastically smaller agent, than in the symmetric variability case. In other words, to Goliath, David is insignificant, but for David, Goliath can be the stimulus of extra efforts.

Social Joy. Throughout the chapter, we assume that the behind-averse parameter dominates the ahead-seeking parameter (i.e., $\ell \geq e$), as stipulated in the prospect theory. This situation is termed *social regret* in Avcı et al. (2014). To some extent, our results may be extended to the other *social joy* case where the ahead-seeking parameter dominates the behind aversion par1ameter (i.e., $e \geq \ell$). The closed

[7]In the extreme that demand shocks are perfectly positively correlated, in equilibrium, there is no profit gap between the two agents, so this case reduces to the deterministic case. In the other extreme that demand shocks are perfectly negatively correlated, the agents' behaviors are qualitatively analogous to the case with independent demand shocks, but with a larger magnitude. Again, by comparing the two extreme cases, it is again confirmed that positive correlation of demand shocks alleviates the variable inequality effect, and negative correlation of demand shocks exacerbates the variable inequality effect.

form expression (5.4), as a solution to the first order conditions of utility functions, may be sustained under additional (and more complex) conditions which guarantee that the utility functions are quasi-concave even for the social joy case. Our results would still hold whenever this form of equilibrium is sustained. In general, the social joy case usually has multiple and asymmetric equilibria (Avcı et al. 2014), which requires a separately rigorous treatment. Our perspective on the variable-inequality effect tends to predict that for the social joy case, when demand variability increases, the effect of a price increase due to ahead seeking would dominate the effect of a price decrease due to behind aversion, leading to less competitive behavior. One implication is that, induced by executive compensation schemes, ahead seeking can be more significant than behind aversion, e.g., when beating competitors is rewarded but being beaten by competitors is not punished. Therefore it may be expected that, even with the co-existence of behind aversion and ahead seeking, social comparison with social joy can result in an overall less intensified price competition, which benefits the agents but hurts consumers.

Other Applications. Price competition among substitutable products is a supermodular game where price decisions are strategic complements. On the other hand, price competition among complementary products is a submodular game where price decisions are strategic substitutes. The managerial insights we have obtained on price competition with social comparisons for substitutable or complementary products also shed light on other games of strategic complements or substitutes if they are played in the presence of social comparison and uncertainty. First, our insights are readily applicable to other games in which the profit functions are quadratic in the decision variables; e.g., see Vilcassim et al. (1999) for an advertising competition game where the profit functions are quadratic in advertising decisions and see Jackson and Zeno (2014) (Section 4.4) for quadratic network games. Second, the expected-comparison effect and the variable-inequality effect should be useful in predicting the competitive behavior in other games under social comparison and uncertainty. The ahead seeking would lead to more weight in the more favorable scenarios in one's mental accounting, and the behind-averse behavior would lead to more weight in the more disadvantageous scenarios, resulting in possibly different reactions. Those reactions by one decision maker may provide an incentive for the competitors to go in the same direction in supermodular games and in the opposite direction for submodular games.

Concluding Remarks. Our results demonstrate how social incentives and demand uncertainty interact to influence agents' competitive behavior in setting prices. In contrast to the deterministic demand case, opposite-directional comparison can have a different effect when there is demand uncertainty. Our model is parsimonious; yet it captures the first-order effects. The analysis is made for fairly generally distributed demand shocks. The effects we found can shed light on other strategic interactions in the presence of social incentives and decision making under uncertainty.

5.7 Appendix

Proof of Corollary 5.1. (i) To prove the result, we first show that $\bar{p} > 0$ and that $\frac{\partial u_i(\bar{p},\bar{p})}{\partial p_i} = 0$.

By defining $r \equiv \frac{1}{2+\ell+e}$, we can rewrite $\bar{p}$ as $\bar{p} = \frac{m+2(\ell-e)r\sigma}{2(1-\gamma r)}$. When $e \geq -1$, we have $2(\ell-e)r\sigma = \frac{2(\ell-e)}{2+\ell+e}\sigma \geq 2\sigma \geq -\frac{\bar{\alpha}}{2} \geq -\frac{m}{2}$. Here, the second inequality holds because σ reaches its minimum $-\frac{\bar{\alpha}}{4}$ when ϵ follows the two-point distribution with $\Pr(\epsilon = -\bar{\alpha}) = \Pr(\epsilon = \bar{\alpha}) = \frac{1}{2}$ (See the discussions after Lemma 5.2). The last one holds because $\bar{\alpha} \leq m$ by assumption. Thus, $\bar{p} > 0$.

Let us solve $\frac{\partial u_1(p,p)}{\partial p_1} = 0$. From (5.2), we can calculate the derivative $\frac{\partial u_1(p_1,p_2)}{\partial p_1} = \frac{\partial EU_1(p_1,p_2)}{\partial p_1}$.

$$\frac{\partial u_1(p_1,p_2)}{\partial p_1} = \gamma p_2 + (1+e)(m-2p_1) + (\ell-e)E\left[(m-2p_1+\epsilon_1)1_{\{p_1(m-p_1+\gamma p_2+\epsilon_1)<p_2(m-p_2+\gamma p_1+\epsilon_2)\}}\right]. \tag{5.10}$$

By setting $p_1 = p_2 = p$ in the above equation, we have

$$\frac{\partial u_1(p,p)}{\partial p_1} = \frac{(2+\ell+e)m + 2(\ell-e)\sigma}{2} - (2+\ell+e-\gamma)p,$$

if $p > 0$. Then, we can obtain the expression of $\bar{p}$ in (5.4) by solving $\frac{\partial u_1(\bar{p},\bar{p})}{\partial p_1} = 0$. Symmetrically, $\frac{\partial u_2(\bar{p},\bar{p})}{\partial p_2} = 0$.

If $\bar{p} \in (0, p^{\max}]$, then it is easy to see from the concavity of $u_i(p_1,p_2)$ that $p_1 = \bar{p}$ is the solution to $\max_{p_1\in[0,p^{\max}]} u_1(p_1,\bar{p})$ and $p_2 = \bar{p}$ solves $\max_{p_1\in[0,p^{\max}]} u_2(\bar{p},p_2)$. If $\bar{p} > p^{\max}$, then it follows from $\frac{\partial u_i(\bar{p},\bar{p})}{\partial p_i} = 0$ that $\frac{\partial u_i(p^{\max},p^{\max})}{\partial p_i} \geq 0$ (from (5.10), we can verify that $\frac{\partial u_i(p,p)}{\partial p_i}$ is decreasing in p), which again implies that $p_i = p^{\max}$ is the best response of agent i to the other agent's price $p_{-i} = p^{\max}$. Therefore, $p_1 = p_2 = p^*$ is an equilibrium.

(ii) We prove the uniqueness of the equilibrium by contradiction. Suppose that there is an equilibrium in which $0 \leq p_2 < p_1$.

Let us consider the function $H(p_1,p_2) \equiv \frac{\partial u_1(p_1,p_2)}{\partial p_1}$. From (5.10), we can write $H(p_1,p_2)$ as

$$H(p_1,p_2) = \gamma p_2 + (1+e)(m-2p_1) + (\ell-e)E_{\epsilon_1}\left[(m-2p_1+\epsilon_1)\bar{F}_{\epsilon_2}\left(\frac{p_1(m-p_1+\epsilon_1)-p_2(m-p_2)}{p_2}\right)\right],$$

where $\bar{F}_{\epsilon_2}(x) = \Pr(\epsilon_2 > x)$. If we denote $g(p_1,p_2,\epsilon_1) = (m-2p_1+\epsilon_1)\bar{F}_{\epsilon_2}\left(\frac{p_1(m-p_1+\epsilon_1)-p_2(m-p_2)}{p_2}\right)$, then $H(p_1,p_2) = \gamma p_2 + (1+e)(m-2p_1) + (\ell-e)E_{\epsilon_1}g(p_1,p_2,\epsilon_1)$.

Since ϵ_1 and ϵ_2 follow i.i.d. distributions, we have

$$\begin{aligned}\frac{\partial u_2(p_1,p_2)}{\partial p_2} &= \gamma p_1 + (1+e)(m-2p_2) + (\ell-e)E_{\epsilon_2}\left[(m-2p_2+\epsilon_2)\bar{F}_{\epsilon_1}\left(\frac{p_2(m-p_2+\epsilon_2)-p_1(m-p_1)}{p_1}\right)\right]\\ &= \gamma p_1 + (1+e)(m-2p_2) + (\ell-e)E_{\epsilon_1}\left[(m-2p_2+\epsilon_1)\bar{F}_{\epsilon_2}\left(\frac{p_2(m-p_2+\epsilon_1)-p_1(m-p_1)}{p_1}\right)\right]\\ &= H(p_2,p_1).\end{aligned}$$

Let us investigate the following difference.

$$\frac{p_1(m-p_1+\epsilon_1)-p_2(m-p_2)}{p_2}-\frac{p_2(m-p_2+\epsilon_1)-p_1(m-p_1)}{p_1}=\frac{(m-p_1-p_2+\epsilon_1)(p_1^2-p_2^2)}{p_1p_2}. \quad (5.11)$$

We see that $\bar{F}_{\epsilon_2}\left(\frac{p_1(m-p_1+\epsilon_1)-p_2(m-p_2)}{p_2}\right) \geq \bar{F}_{\epsilon_2}\left(\frac{p_2(m-p_2+\epsilon_2)-p_1(m-p_1)}{p_1}\right)$ if $\epsilon_1 \leq p_1+p_2-m$, and that $\bar{F}_{\epsilon_2}\left(\frac{p_1(m-p_1+\epsilon_1)-p_2(m-p_2)}{p_2}\right) \leq \bar{F}_{\epsilon_2}\left(\frac{p_2(m-p_2+\epsilon_2)-p_1(m-p_1)}{p_1}\right)$ if $\epsilon_1 \geq p_1+p_2-m$.

Next, we compare $g(p_1,p_2,\epsilon_1)$ and $g(p_2,p_1,\epsilon_2)$ by three cases.

Case 1: $\epsilon_1 \leq p_1+p_2-m$.

In this case, we have $m-2p_1+\epsilon_1 < m-p_1-p_2+\epsilon_1 \leq 0$. If $m-2p_2+\epsilon_1 \geq 0$, then $g(p_1,p_2,\epsilon_1) \leq 0 \leq g(p_2,p_1,\epsilon_1)$. Otherwise, $m-2p_1+\epsilon_1 < m-2p_2+\epsilon_1 < 0$. Together with the result $\bar{F}_{\epsilon_2}\left(\frac{p_1(m-p_1+\epsilon_1)-p_2(m-p_2)}{p_2}\right) \geq \bar{F}_{\epsilon_2}\left(\frac{p_2(m-p_2+\epsilon_2)-p_1(m-p_1)}{p_1}\right) \geq 0$, again we have $g(p_1,p_2,\epsilon_1) \leq g(p_2,p_1,\epsilon_1)$.

Case 2: $p_1+p_2-m < \epsilon_1 \leq 2p_1-m$.

In this case, we have $m-2p_2+\epsilon_1 > m-p_1-p_2+\epsilon_1 > 0$ and $m-2p_1+\epsilon_1 \leq 0$. Then, $g(p_1,p_2,\epsilon_1) \leq 0 \leq g(p_2,p_1,\epsilon_1)$.

Case 3: $\epsilon_1 > 2p_1-m$.

In this case, we have $m-2p_2+\epsilon_1 > m-2p_1+\epsilon_1 > 0$ and $\epsilon_1 > p_1+p_2-m$. The latter inequality implies $\bar{F}_{\epsilon_2}\left(\frac{p_1(m-p_1+\epsilon_1)-p_2(m-p_2)}{p_2}\right) \leq \bar{F}_{\epsilon_2}\left(\frac{p_2(m-p_2+\epsilon_2)-p_1(m-p_1)}{p_1}\right)$. It follows that $g(p_1,p_2,\epsilon_1) \leq g(p_2,p_1,\epsilon_1)$.

Combining the above three cases, we can conclude that $E_{\epsilon_1}g(p_1,p_2,\epsilon_1) \leq E_{\epsilon_1}g(p_2,p_1,\epsilon_1)$. Then,

$$H(p_1,p_2)-H(p_2,p_1) = -\left[\gamma+2(1+e)\right](p_1-p_2)+(\ell-e)\left[E_{\epsilon_1}g(p_1,p_2,\epsilon_1)-E_{\epsilon_1}g(p_2,p_1,\epsilon_1)\right] < 0. \quad (5.12)$$

It is easy to verify that $\frac{\partial u_2(p_1,0)}{\partial p_2} > 0$, implying that the expected utility can be improved if p_2 is increased by a small amount. Thus, $p_2 > 0$.

If $p_1 < p^{\max}$, then $\frac{\partial u_i(p_1,p_2)}{\partial p_i} = 0$ for $i=1,2$, or equivalently, $H(p_1,p_2)=H(p_2,p_1)=0$, which contradicts (5.12).

If $p_1 = p^{\max}$, then $H(p_1,p_2) = \frac{\partial u_1(p_1,p_2)}{\partial p_1} \geq 0$. Consequently, we have $\frac{\partial u_2(p_1,p_2)}{\partial p_2} = H(p_2,p_1) > H(p_1,p_2) \geq 0$. Then, agent 2's expected utility $u_2(p_1,p_2)$ can be improved if p_2 is slightly increased (note that $p_2 < p_1 = p^{\max}$), contradicting the optimality of p_2 for the given p_1.

Therefore, we proved that the prices with $p_2 < p_1$ cannot be an equilibrium. Symmetrically, we can show that the prices p_1 and p_2 with $p_1 < p_2$ do not constitute an equilibrium. □

Proof of Corollary 5.4. (i) Price. We can calculate that

$$\frac{\mathrm{d}\bar{p}}{\mathrm{d}\ell} = \frac{2(2+2e-\gamma)\sigma-\gamma m}{2(2+\ell+e-\gamma)^2}.$$

Note that $\gamma < 0$. Then, $\frac{\mathrm{d}\bar{p}}{\mathrm{d}\ell} \geq 0$ if $\sigma \geq \frac{\gamma m}{2(2+2e-\gamma)}$ and $\frac{\mathrm{d}\bar{p}}{\mathrm{d}\ell} \leq 0$ otherwise. As a result, $\bar{p}$ (thus also p^*) is increasing in ℓ if the market variability is below the threshold at which $\sigma = \hat{\sigma}_1(e) \equiv \frac{\gamma m}{2(2+2e-\gamma)}$ and is decreasing in ℓ otherwise.

PROFIT. Note that $\pi(p,p)$ is concave in p and reaches its maximum at $\widetilde{p} = \frac{m}{2(1-\gamma)}$. A simple calculation reveals that $\bar{p} > \widetilde{p}$ if and only if $\sigma > \hat{\sigma}_2(\ell,e) \equiv \frac{\gamma(1+\ell+e)}{2(1-\gamma)(\ell-e)}m$. This implies that $\pi(\bar{p},\bar{p})$ is decreasing in $\bar{p}$ if if $\sigma > \check{\sigma}(\ell,e)$ and is increasing in $\bar{p}$ otherwise.

To proceed further, we compare the two thresholds $\hat{\sigma}_1(e)$ and $\hat{\sigma}_2(\ell,e)$.

$$\hat{\sigma}_2(\ell,e) - \hat{\sigma}_1(e) = \frac{\gamma(1+\ell+e)m}{2(1-\gamma)(\ell-e)} - \frac{\gamma m}{2(2+2e-\gamma)} = \frac{\gamma m(1+2e)(\ell+e+2-\gamma)}{2(1-\gamma)(\ell-e)(2+2e-\gamma)},$$

from which we see that $\hat{\sigma}_1(e) \leq \hat{\sigma}_2(\ell,e)$ if and only if $e \leq -\frac{1}{2}$.

First, let us consider the case where $e < -\frac{1}{2}$. It is easy to see that $\hat{\sigma}_2(\ell,e)$ is decreasing in ℓ in this case. For a given σ, let $L(\sigma,e)$ be a threshold value of ℓ such that $\sigma < \hat{\sigma}_2(\ell,e)$ for $\ell < L(\sigma,e)$ and $\sigma \geq \hat{\sigma}_2(\ell,e)$ otherwise. More specifically, we can show that

$$L(\sigma,e) = \begin{cases} -\infty & \text{if } \sigma \geq \hat{\sigma}_2(0,e), \\ \frac{\gamma(1+2e)m}{2(1-\gamma)\sigma-\gamma m} + e & \text{if } \hat{\sigma}_2(\infty,e) \leq \sigma < \hat{\sigma}_2(0,e), \\ \infty & \text{if } \sigma < \hat{\sigma}_2(\infty,e). \end{cases}$$

Suppose that $\sigma > \hat{\sigma}_1(e)$. If $\ell < L(\sigma,e)$, then $\hat{\sigma}_1(e) \leq \sigma < \hat{\sigma}_2(\ell,e)$. Consequently, $\bar{p}$ is increasing in ℓ and $\bar{p} < \widetilde{p}$. Thus, $\pi(\bar{p},\bar{p})$, as well as $\pi(p^*,p^*)$, is increasing in ℓ. If $\ell \geq L(\sigma,e)$, then $\hat{\sigma}_1(e) \leq \hat{\sigma}_2(\ell,e) \leq \sigma$. In this case, $\bar{p}$ is still increasing in ℓ but $\bar{p} \geq \widetilde{p}$, implying that $\pi(p^*,p^*)$ is decreasing in ℓ.

If $\sigma \leq \hat{\sigma}_1(e)$, we have $\sigma \leq \hat{\sigma}_1(e) \leq \hat{\sigma}_2(\ell,e)$. In this case, $\bar{p}$ is decreasing in ℓ and $\bar{p} \leq \widetilde{p}$. Therefore, $\pi(p^*,p^*)$ is decreasing in ℓ.

Thus, we have proved that $\pi(p^*,p^*)$ is quasi-concave in ℓ if $e < -\frac{1}{2}$.

Similarly, we can show that $\pi(p^*,p^*)$ is quasi-concave in ℓ if $e \geq -\frac{1}{2}$, with the following changeover point:

$$\widetilde{L}(\sigma,e) = \begin{cases} -\infty & \text{if } \sigma < \hat{\sigma}_2(e^+,e) \text{ or } \sigma > \hat{\sigma}_1(e), \\ \frac{\gamma(1+2e)m}{2(1-\gamma)\sigma-\gamma m} + e & \text{if } \hat{\sigma}_2(e^+,e) \leq \sigma \leq \hat{\sigma}_2(\infty,e), \\ \infty & \text{if } \hat{\sigma}_2(\infty,e) < \sigma \leq \hat{\sigma}_1(e), \end{cases}$$

where $e^+ = \max\{0,e\}$.

(ii) PRICE. Let us calculate the derivative of $\bar{p}$ with respect to e.

$$\frac{\mathrm{d}\bar{p}}{\mathrm{d}e} = -\frac{2(2+2\ell-\gamma)\sigma+\gamma m}{2(2+\ell+e-\gamma)^2},$$

from which we see that $\bar{p}$, thus also p^*, increases in e.

PROFIT. As shown in (i), we have $\bar{p} < \widetilde{p}$ if and only if $\sigma < \hat{\sigma}_2(\ell,e) \equiv \frac{\gamma(1+\ell+e)}{2(1-\gamma)(\ell-e)}$.

It is easy to see that $\hat{\sigma}_2(\ell, e)$ is decreasing in e. Then, there exists $\varepsilon(\sigma, \ell)$ (possibly equal to ∞ or $-\infty$), such that $\sigma < \hat{\sigma}(\ell, e)$ if $e < \varepsilon(\sigma, \ell)$ and $\sigma \geq \hat{\sigma}(\ell, e)$ otherwise. In particular,

$$\varepsilon(\sigma, \ell) = \begin{cases} \infty & \text{if } \sigma < \hat{\sigma}_2(\ell, \ell), \\ \ell - \frac{\gamma m(1+2\ell)}{2(1-\gamma)\sigma + \gamma m} & \text{if } \hat{\sigma}_2(\ell, \ell) \leq \sigma < \hat{\sigma}_2(\ell, -\min\{\ell, 1\}), \\ -\infty & \text{if } \sigma \geq \hat{\sigma}_2(\ell, -\min\{\ell, 1\}). \end{cases}$$

It then follows that $\pi(p^*, p^*)$ is quasi-concave with $\varepsilon(\sigma, \ell)$ being the changeover point. □

Proof of Corollary 5.5. (i) PRICE. Let us start with calculating the derivative $\partial EU_i(p_1, p_2, \boldsymbol{\epsilon})/\partial p_i$.

$$\begin{aligned} &\frac{\partial EU_i(p_1, p_2, \boldsymbol{\epsilon})}{\partial p_i} \\ =&(1+e)[d_i(p_1, p_2) + p_i \frac{\partial d_i(p_1, p_2)}{\partial p_i}] - e[p_{-i} \frac{\partial d_{-i}(p_1, p_2)}{\partial p_i}] \\ &+ (\ell - e) \left\{ \left[d_i(p_1, p_2) + \epsilon_i + p_i \frac{\partial d_i(p_1, p_2)}{\partial p_i} - p_{-i} \frac{\partial d_{-i}(p_1, p_2)}{\partial p_i} \right] I_{\{p_i[d_i(p_1,p_2)+\epsilon_i] < p_{-i}[d_{-i}(p_1,p_2)+\epsilon_{-i}]\}} \right\}. \end{aligned}$$

If we set $p_1 = p_2 = p$, then

$$\begin{aligned} \frac{\partial EU_i(p, p, \boldsymbol{\epsilon})}{\partial p_i} =&(1+e)d_i(p,p) + (1+e)p\frac{\partial d_i(p,p)}{\partial p_i} - ep\frac{\partial d_{-i}(p,p)}{\partial p_i} \\ &+ (\ell - e)\left\{[d_i(p,p) + p\frac{\partial d_i(p,p)}{\partial p_i} - p\frac{\partial d_i(p,p)}{\partial p_{-i}} + \epsilon_i]I_{\{\epsilon_i < \epsilon_{-i}\}}\right\} \\ =&d_i(p,p) + p\frac{\partial d_i(p,p)}{\partial p_i} + \frac{\ell + e}{2}[d_i(p,p) + p\frac{\partial d_i(p,p)}{\partial p_i} - p\frac{\partial d_{-i}(p,p)}{\partial p_i}] + (\ell - e)E[\epsilon_i I_{\{\epsilon_i < \epsilon_{-i}\}}]. \end{aligned}$$

For ease of notation, we write $g(p) = d_i(p,p) + p\partial d_i(p,p)/\partial p_i$ and $h(p) = d_i(p,p) + p\partial d_i(p,p)/\partial p_i - p\partial d_{-i}(p,p)/\partial p_i$. By assumption, both $g(p)$ and $h(p)$ are decreasing in p, thus so is $\partial EU_i(p,p,\boldsymbol{\epsilon})/\partial p_i = g(p) + (\ell+e)h(p)/2 + (\ell-e)\sigma$. We can rewrite $\partial EU_i(p,p,\boldsymbol{\epsilon})/\partial p_i$ as:

$$\frac{\partial EU_i(p, p, \boldsymbol{\epsilon})}{\partial p_i} = g(p) + e \cdot h(p) + (\ell - e)[\frac{1}{2}h(p) + \sigma].$$

The symmetric equilibrium $p^*(\ell, e, \sigma)$, as a function of (ℓ, e, σ), is determined by

$$p^*(\ell, e, \sigma) = \inf\left\{p \in [0, p^{\max}] \mid g(p) + e \cdot h(p) + (\ell - e)[\frac{1}{2}h(p) + \sigma] \leq 0\right\}. \tag{5.13}$$

In other words, $p^*(\ell, e, \sigma)$ is the smallest $p \in [0, p^{\max}]$ such that either $g(p) + e \cdot h(p) + (\ell - e)[\frac{1}{2}h(p) + \sigma]$ hits 0 for the first time, or $g(p^{\max}) + e \cdot h(p^{\max}) + (\ell - e)[\frac{1}{2}h(p^{\max}) + \sigma] > 0$ and $p^* = p^{\max}$. It is easy to see that $p^*(\ell, e, \sigma)$ is increasing in σ. In fact, for $\sigma' > \sigma$, we have $g(p^*(\ell, e, \sigma)) + e \cdot h(p^*(\ell, e, \sigma)) + (\ell - e)[\frac{1}{2}h(p^*(\ell, e, \sigma)) + \sigma'] \geq g(p^*(\ell, e, \sigma)) + e \cdot h(p^*(\ell, e, \sigma)) + (\ell - e)[\frac{1}{2}h(p^*(\ell, e, \sigma)) + \sigma] \geq 0$. This implies that $p^*(\ell, e, \sigma') \geq p^*(\ell, e, \sigma)$.

As σ increases, $p^*(\ell,e,\sigma)$ increases, causing both $g(p^*)$ and $h(p^*)$ to decrease. Before $p^*(\ell,e,\sigma)$ reaches $p^{\max}$, $g(p^*)+e\cdot h(p^*)+(\ell-e)[\frac{1}{2}h(p^*)+\sigma]=0$. An increasing p^* with respect to σ implies a decreasing $g(p^*)+e\cdot h(p^*)$. Then, $\frac{1}{2}h(p^*)+\sigma=-[g(p^*)+e\cdot h(p^*)]/(\ell-e)$ is increasing in σ. If p^* reaches $p^{\max}$, it will remain $p^{\max}$ thereafter, in which case $\frac{1}{2}h(p^*)+\sigma=\frac{1}{2}h(p^{\max})+\sigma$ continues to increase in σ.

Thus $\frac{1}{2}h(p^*)+\sigma$ is increasing in σ. Upon reaching $p^{\max}$, p^* remains constant, in which case $\frac{1}{2}h(p^*)+\sigma$ still increases in σ.

Next, we argue that $\frac{1}{2}h(p^*)+\sigma\le 0$. To this end, let us suppose to the contrary that $\frac{1}{2}h(p^*)+\sigma>0$. Since $\sigma\le 0$ and $\partial d_{-i}(p,p)/\partial p_i\ge 0$, $g(p^*)=h(p^*)+p\partial d_{-i}(p^*,p^*)/\partial p_i\ge h(p^*)\ge h(p^*)+2\sigma>0$. As a result, $g(p^*)+e\cdot h(p^*)+(\ell-e)[\frac{1}{2}h(p^*)+\sigma]>0$, contradicting (5.13).

It then follows the nonpositiveness of $\frac{1}{2}h(p^*)+\sigma$ that for $\ell'>\ell$,

$$\begin{aligned}&g(p^*(\ell,e,\sigma))+e\cdot h(p^*(\ell,e,\sigma))+(\ell'-e)[\frac{1}{2}h(p^*(\ell,e,\sigma))+\sigma]\\ \le&g(p^*(\ell,e,\sigma))+e\cdot h(p^*(\ell,e,\sigma))+(\ell-e)[\frac{1}{2}h(p^*(\ell,e,\sigma))+\sigma]=0,\end{aligned}$$

if $p^*(\ell,e,\sigma)<p^{\max}$. Thus, $p(\ell',e,\sigma)\le p^*(\ell,e,\sigma)$. If $p^*(\ell,e,\sigma)=p^{\max}$, then $p(\ell',e,\sigma)\le p^*(\ell,e,\sigma)$ holds trivially.

Profit. Under the symmetric equilibrium, agent i's expected profit is $\pi_i(p^*,p^*)=p^*d_i(p^*,p^*)$. Let $\hat{p}$ maximizes $\pi(p,p)$ with respect to $p\in[0,\infty)$. Then $\pi_i(p,p)$ increases in p if $p\le\hat{p}$ and decreases in p if $p>\hat{p}$. The first-order condition requires that $d_i(\hat{p},\hat{p})+\hat{p}(\partial d_i(\hat{p},\hat{p})/\partial p_i+\partial d_i(\hat{p},\hat{p})/\partial p_{-i})=0$. By symmetry, $\partial d_i(\hat{p},\hat{p})/\partial p_{-i}=\partial d_{-i}(\hat{p},\hat{p})/\partial p_i$. Thus $g(\hat{p})=d_i(\hat{p},\hat{p})+\hat{p}\partial d_i(\hat{p},\hat{p})/\partial p_i=-\hat{p}\partial d_{-i}(\hat{p},\hat{p})/\partial p_i$ and $h(\hat{p})=d_i(\hat{p},\hat{p})+\hat{p}\partial d_i(\hat{p},\hat{p})/\partial p_i-\hat{p}\partial d_{-i}(\hat{p},\hat{p})/\partial p_i=-2\partial d_{-i}(\hat{p},\hat{p})/\partial p_i$. It follows that

$$\frac{\partial EU_i(\hat{p},\hat{p})}{\partial p_i}=-(1+\ell+e)\hat{p}\frac{\partial d_{-i}(\hat{p},\hat{p})}{\partial p_i}+(\ell-e)\sigma\le 0.$$

Consequently, $p^*\le\hat{p}$, which implies that the expected profit $\pi_i(p^*,p^*)$ is increasing in p^*. Therefore, $\pi_i(p^*,p^*)$ is decreasing in ℓ.

(ii) Price. We can rewrite

$$\frac{\partial EU_i(p,p,\epsilon)}{\partial p_i}=g(p)+\ell[\frac{1}{2}h(p)+\sigma]+e[\frac{1}{2}h(p)-\sigma].$$

The monotonicity of $p^*(\ell,e,\sigma)$ with respect to e in a small neighborhood of $\mathcal{N}_e\ni e$ is determined by the sign of $\frac{1}{2}h(p^*(\ell,e,\sigma)-\sigma$. If $\frac{1}{2}h(p^*(\ell,e,\sigma)-\sigma>0$, then for $e'>e$,

$$\begin{aligned}&g(p^*(\ell,e,\sigma))+\ell[\frac{1}{2}h(p^*(\ell,e,\sigma))+\sigma]+e'[\frac{1}{2}h(p^*(\ell,e,\sigma))-\sigma]\\ >&g(p^*(\ell,e,\sigma))+\ell[\frac{1}{2}h(p^*(\ell,e,\sigma))+\sigma]+e[\frac{1}{2}h(p^*(\ell,e,\sigma))-\sigma]\ge 0,\end{aligned}$$

implying that $p^*(\ell, e', \sigma) \geq p^*(\ell, e, \sigma)$. Similarly, we can show that $p^*(\ell, e', \sigma) \leq p^*(\ell, e, \sigma)$ if $\frac{1}{2}h(p^*(\ell, e, \sigma) - \sigma < 0$, and $p^*(\ell, e', \sigma) = p^*(\ell, e, \sigma)$ if $\frac{1}{2}h(p^*(\ell, e, \sigma) - \sigma = 0$.

On the other hand, since $p^*(\ell, e, \sigma)$ is increasing in σ and $h(p)$ is decreasing in p, the term $\frac{1}{2}h(p^*(\ell, e, \sigma)) - \sigma$ is decreasing in σ. Consider the case in which $e = 0$. Let $\hat{\sigma}$ be the threshold such that $\frac{1}{2}h(p^*(\ell, 0, \sigma)) - \sigma > 0$ for $\sigma < \hat{\sigma}$ and $\frac{1}{2}h(p^*(\ell, 0, \sigma)) - \sigma < 0$ for $\sigma > \hat{\sigma}$.

Then, if $\sigma < \hat{\sigma}$, $p^*(\ell, e, \sigma)$ will increase in e near $e = 0$. As $p^*(\ell, e, \sigma)$ increases, the value of $\frac{1}{2}h(p^*(\ell, e, \sigma)) - \sigma$ will decrease. Before $\frac{1}{2}h(p^*(\ell, e, \sigma)) - \sigma$ decreases to 0, $p^*(\ell, e, \sigma)$ increases; after $\frac{1}{2}h(p^*(\ell, e, \sigma)) - \sigma$ hits 0 (if ever), $p^*(\ell, e, \sigma)$ will remain a constant.

If $\sigma > \hat{\sigma}$, $p^*(\ell, e, \sigma)$ will decrease in e near $e = 0$. As $p^*(\ell, e, \sigma)$ decreases, the value of $\frac{1}{2}h(p^*(\ell, e, \sigma)) - \sigma$ will increase. $p^*(\ell, e, \sigma)$ will be decreasing in e before $\frac{1}{2}h(p^*(\ell, e, \sigma)) - \sigma$ increases to 0, and remain a constant thereafter.

PROFIT. As shown in (i), the expected profit $\pi_i(p^*, p^*)$ is increasing in p^*. Thus $\pi_i(p^*, p^*)$ is increasing in e if $\sigma < \hat{\sigma}$, and is decreasing in e if $\sigma > \hat{\sigma}$. □

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